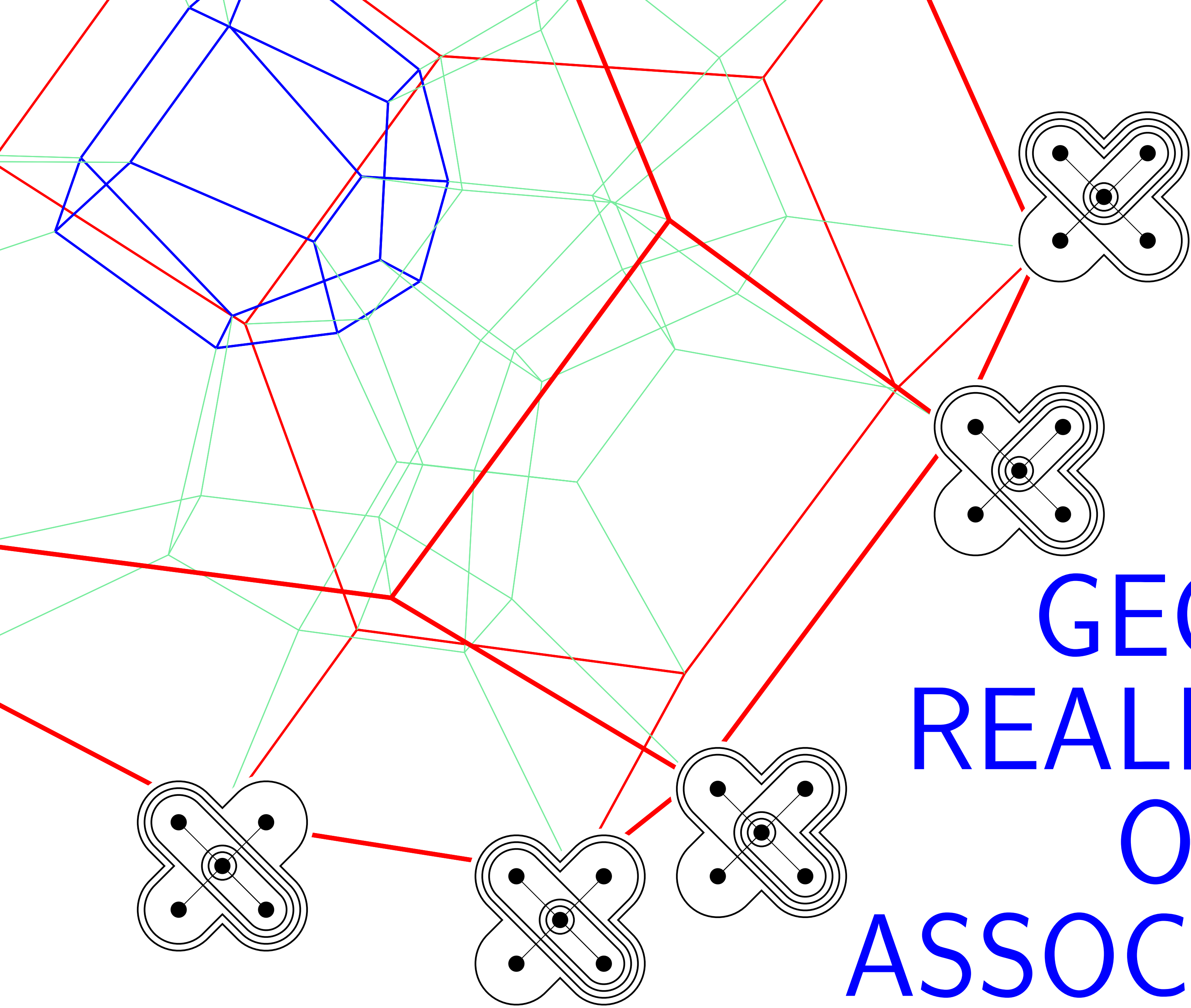


Sage Days 64.5
June 3, 2015

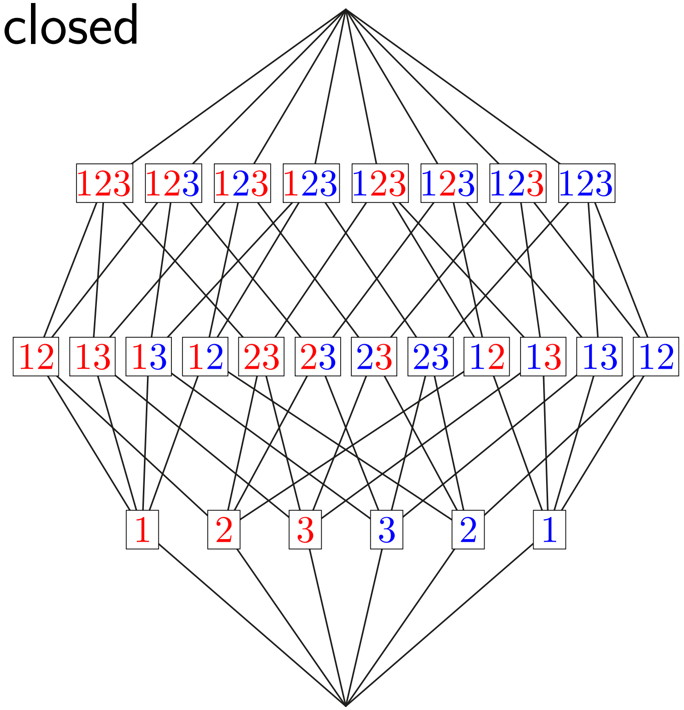
MANY GEOMETRIC REALIZATIONS OF GRAPH ASSOCIAHEDRA

T. MANNEVILLE (LIX)
V. PILAUD (CNRS & LIX)

POLYTOPES & COMBINATORICS

SIMPLICIAL COMPLEX

simplicial complex $=$ collection of subsets of X downward closed



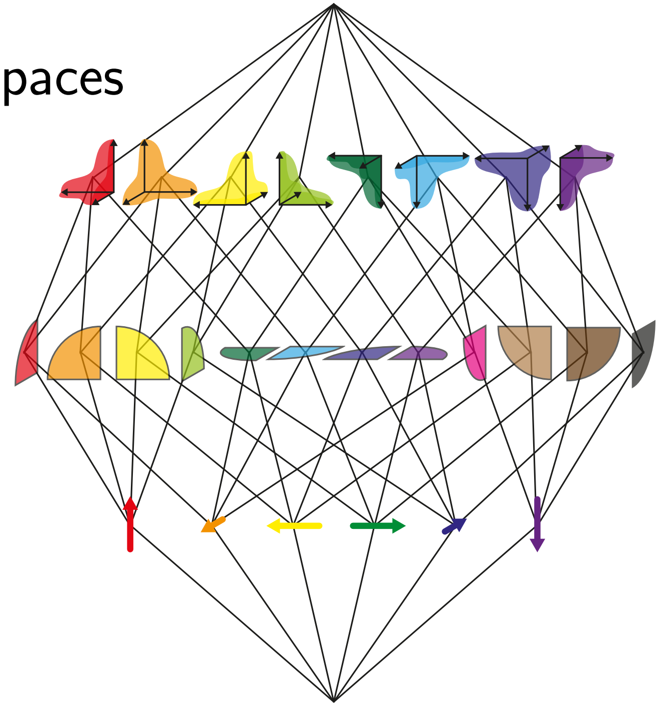
exm:

$$X = [n] \cup [n]$$

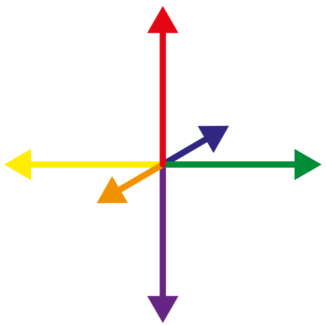
$$\Delta = \{I \subseteq X \mid \forall i \in [n],\ \{i, i\} \not\subseteq I\}$$

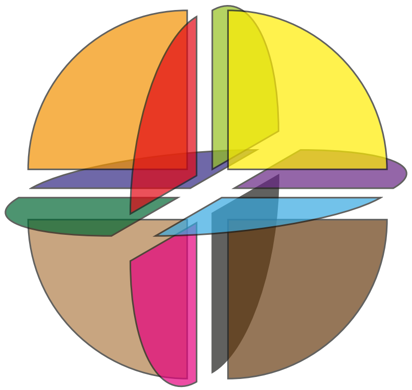
FANS

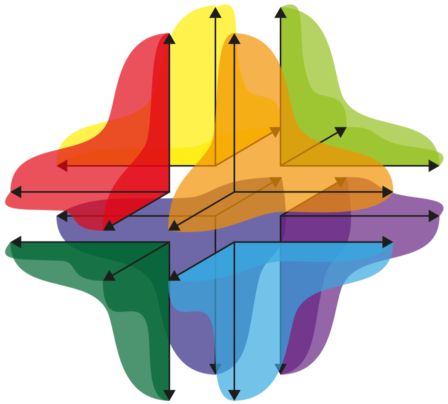
polyhedral cone = positive span of a finite set of $\mathbb{R}^d$
= intersection of finitely many linear half-spaces



fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face







simplicial fan = maximal cones generated by d rays

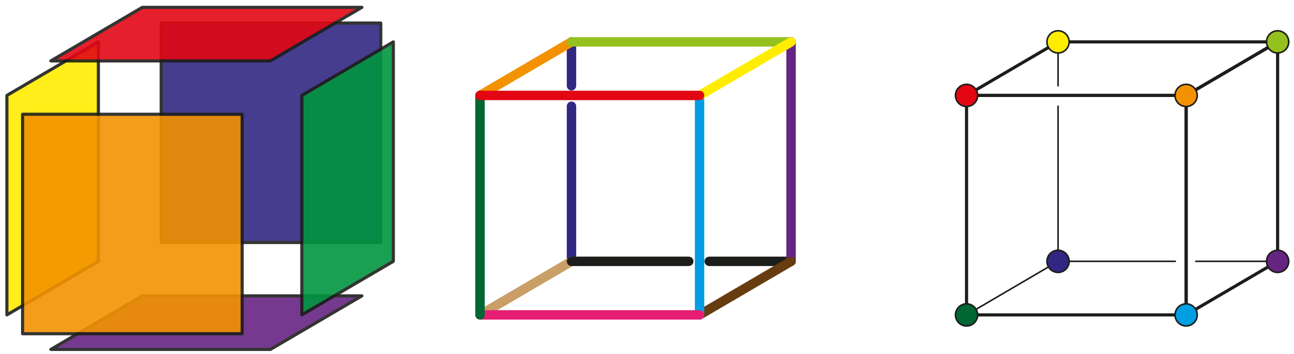
POLYTOPES

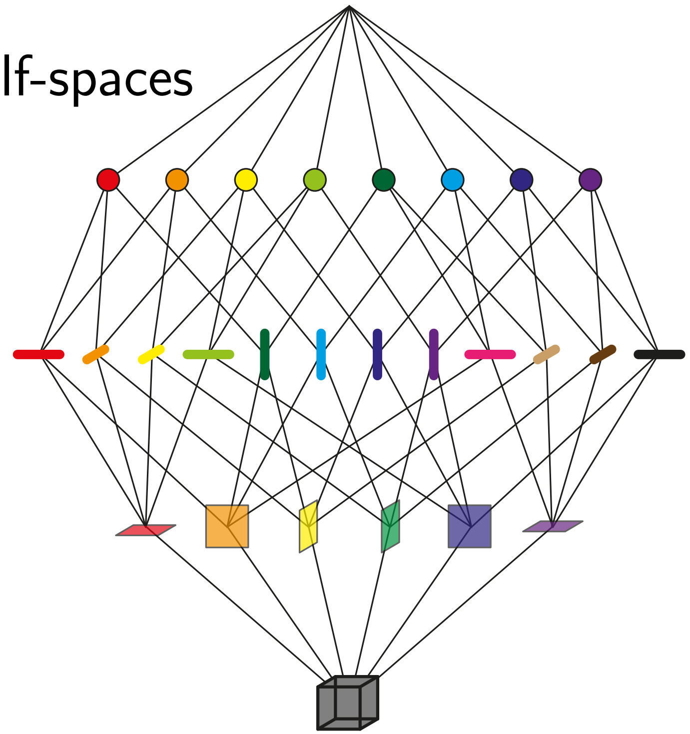
polytope $=$ convex hull of a finite set of $\mathbb{R}^d$

$=$ bounded intersection of finitely many affine half-spaces

face $=$ intersection with a supporting hyperplane

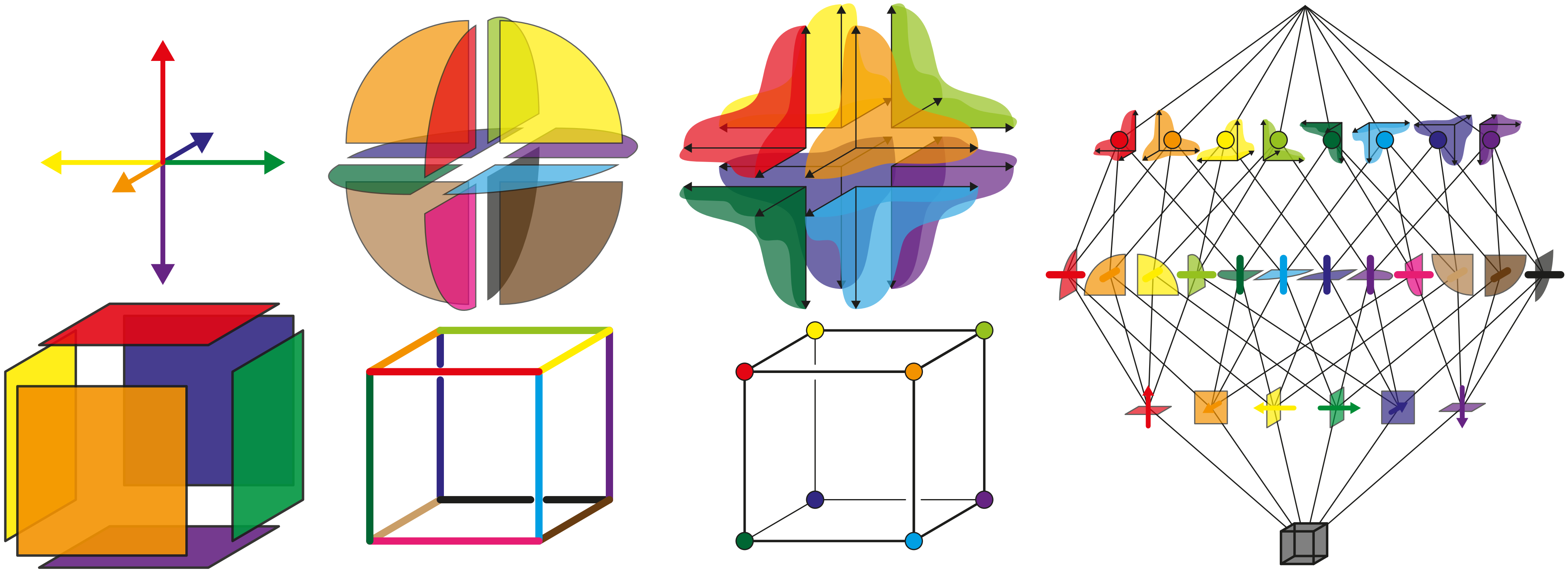
face lattice $=$ all the faces with their inclusion relations





simple polytope $=$ facets in general position $=$ each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



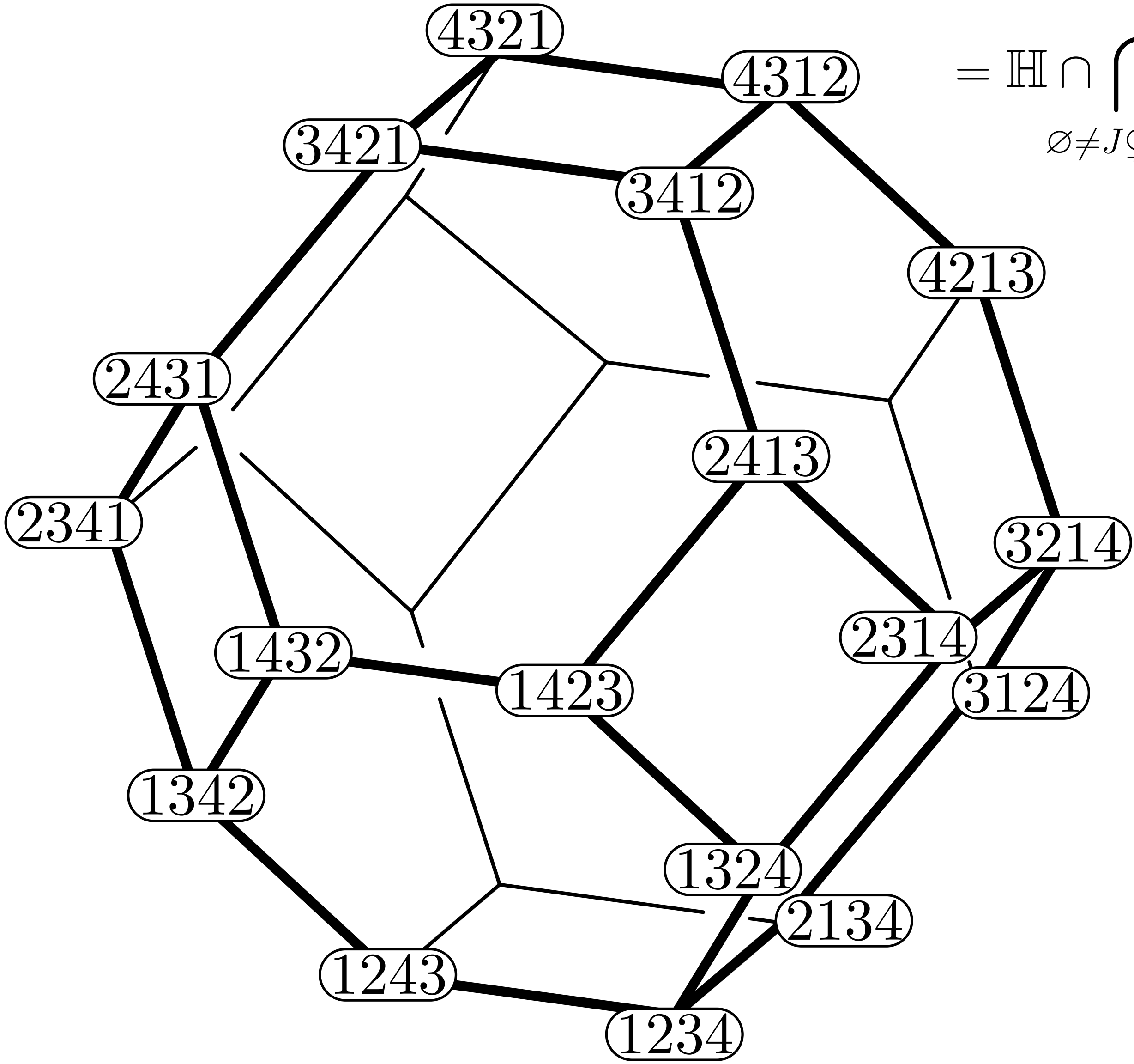
P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

normal fan of P = $\{$ normal cone of F $\mid$ F face of P $\}$

simple polytope $\implies$ simplicial fan $\implies$ simplicial complex

PERMUTAHEDRON

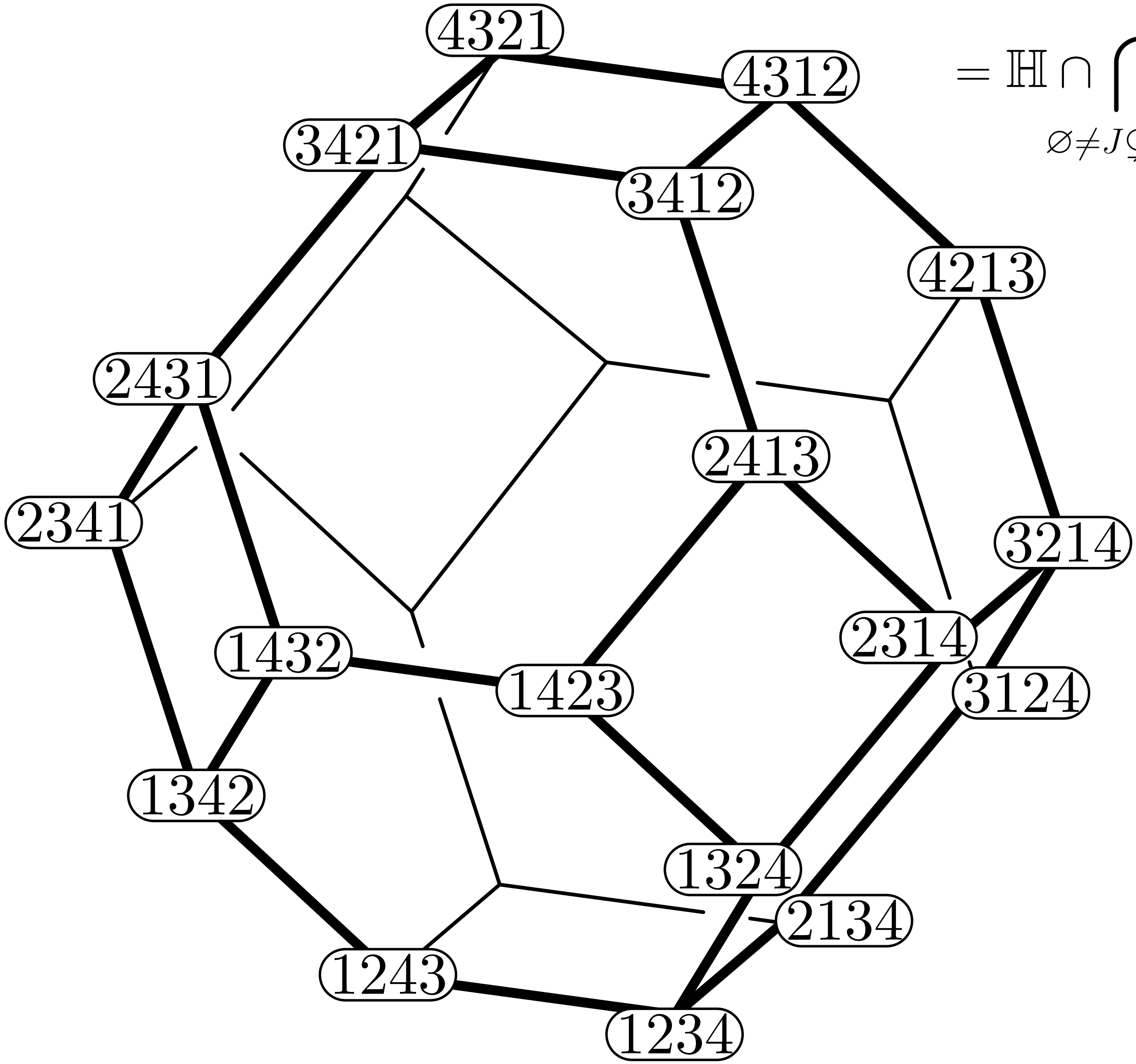


Permutohedron Perm(n)

$$= \operatorname{conv}\left\{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

PERMUTAHEDRON



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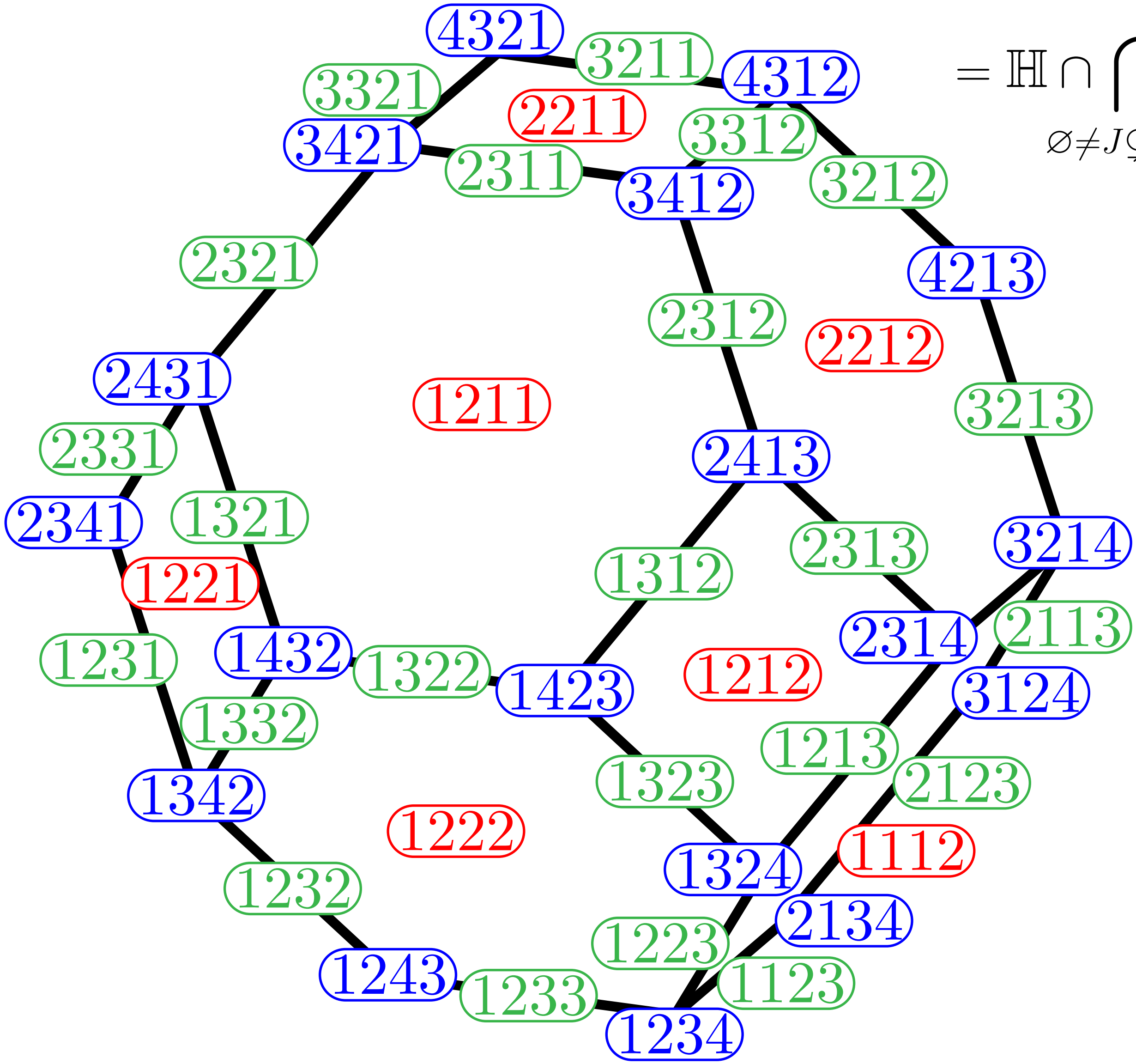
connections to

- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

PERMUTAHEDRON



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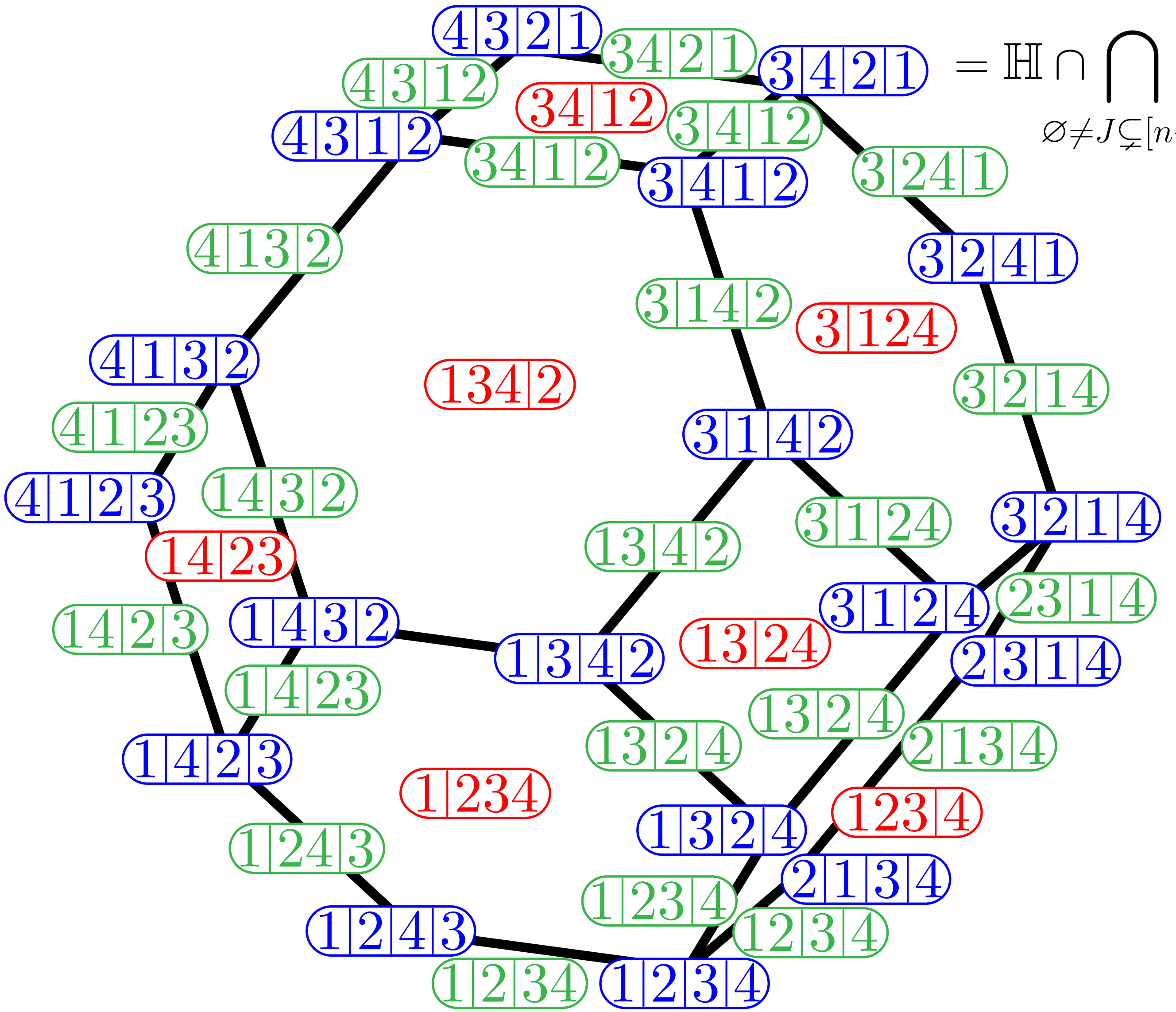
k-faces of Perm(n)

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

PERMUTAHEDRON



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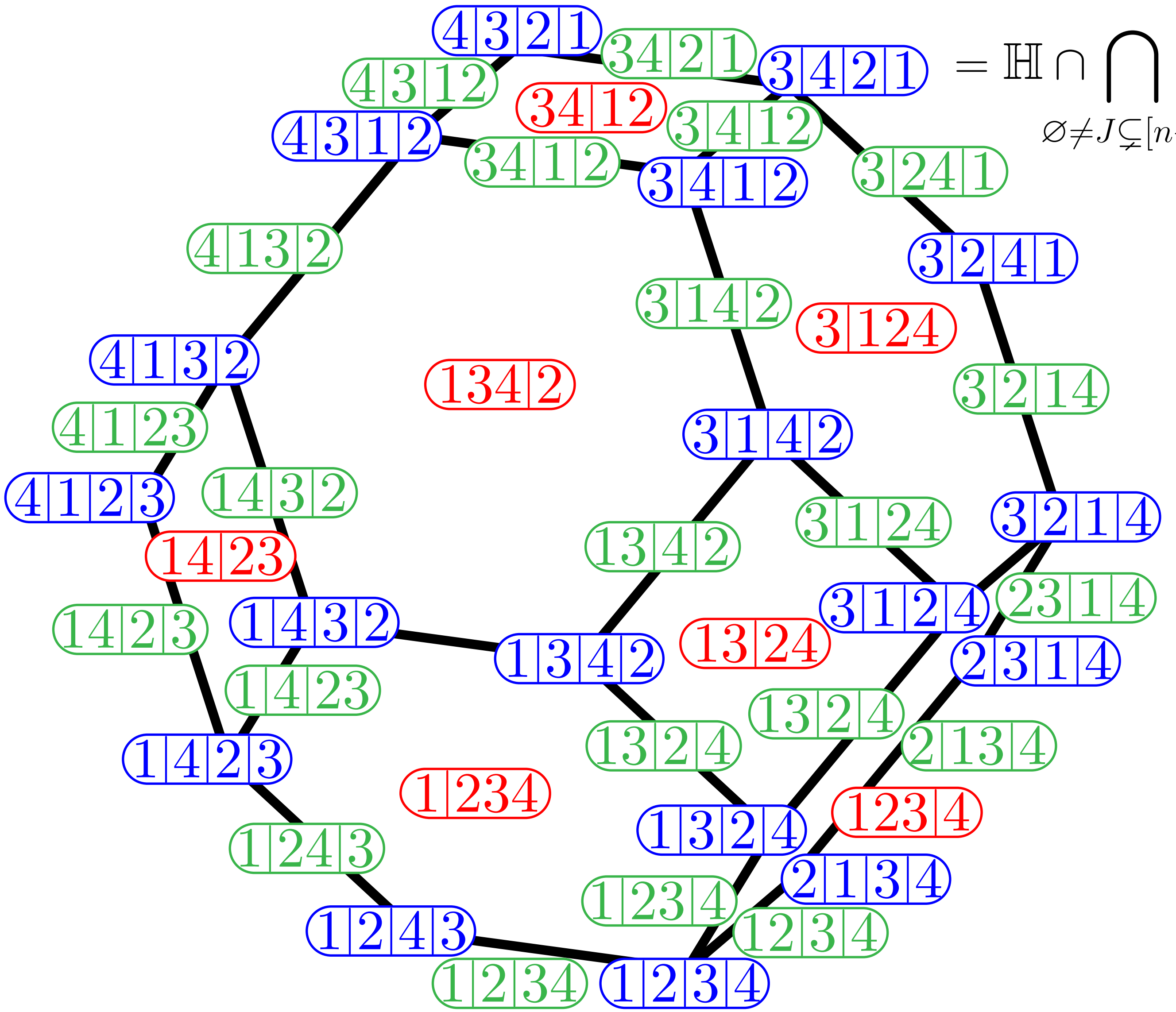
$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

PERMUTAHEDRON



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connections to

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k-faces of Perm(n)

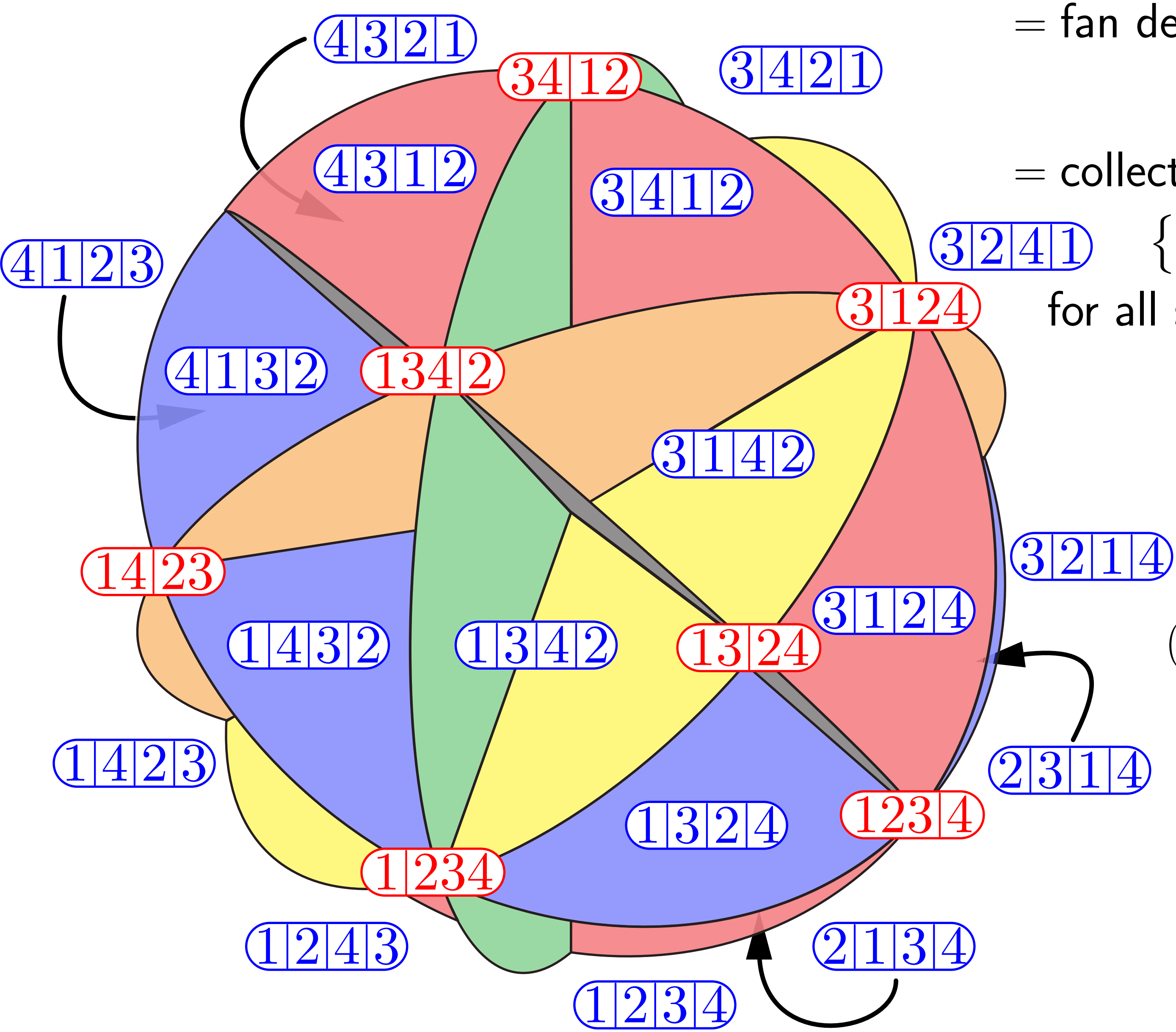
$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

$\equiv$ collections of $n-k$ nested subsets of $[n+1]$

COXETER ARRANGEMENT

Coxeter fan



= fan defined by the hyperplane arrangement

$$\left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; x_i = x_j \right\}_{1 \le i < j \le n+1}$$

= collection of all cones

$$\left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; x_i < x_j \text{ if } \pi(i) < \pi(j) \right\}$$

for all surjections $\pi : [n+1] \to [n+1-k]$

$(n-k)$-dimensional cones

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

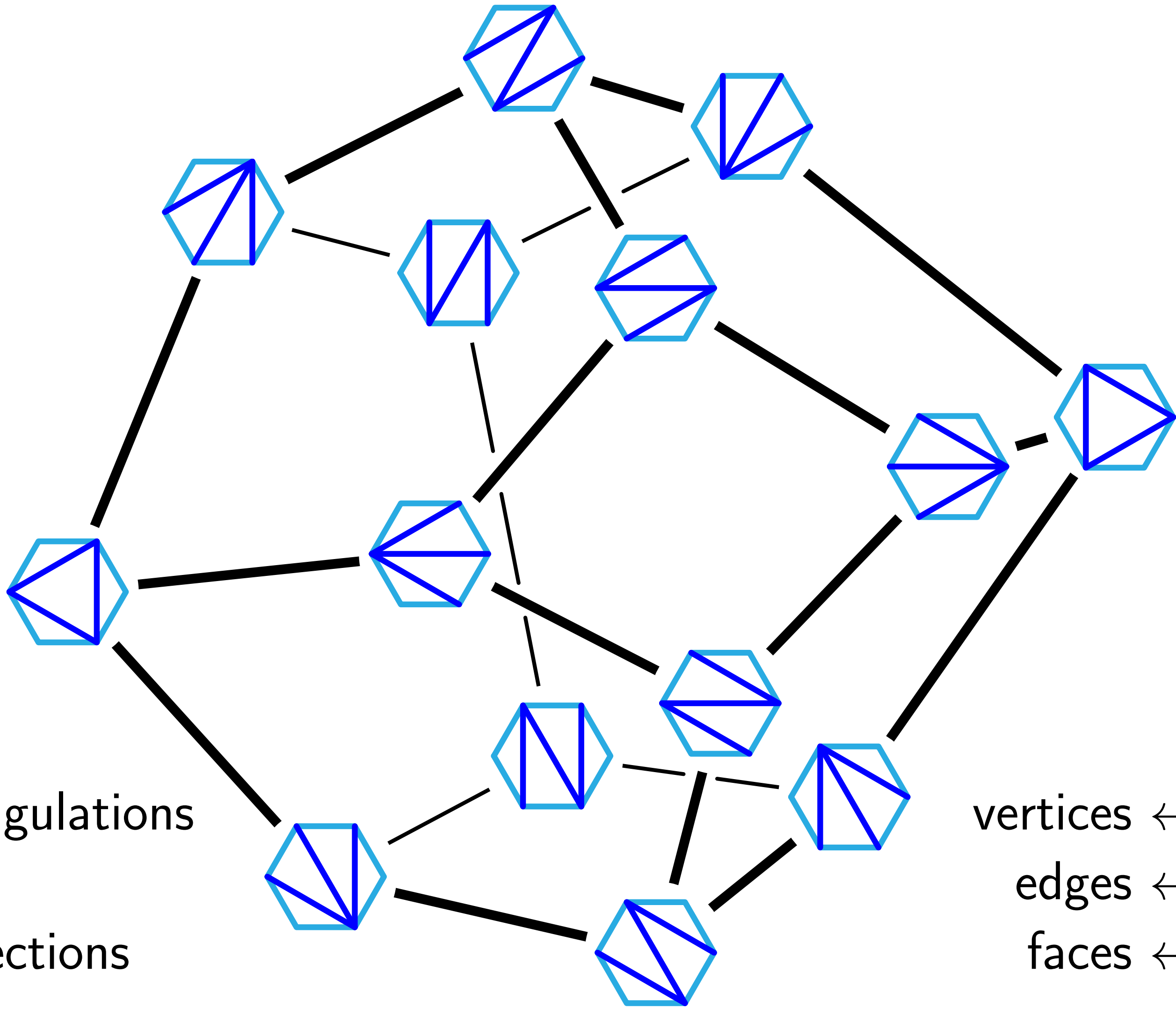
$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

$\equiv$ collections of $n-k$ nested subsets of $[n+1]$

ASSOCIAHEDRA

ASSOCIAHEDRON

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

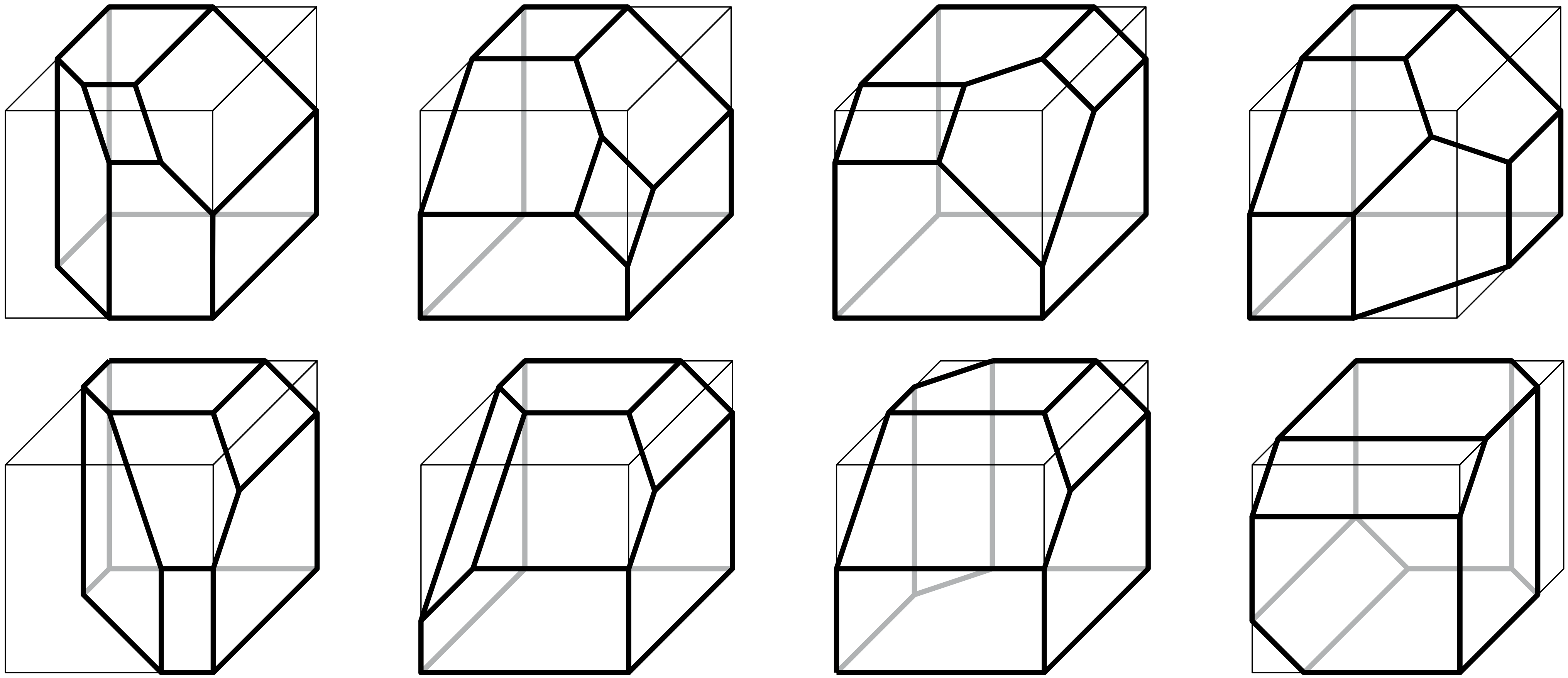


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion



(Pictures by Ceballos-Santos-Ziegler)

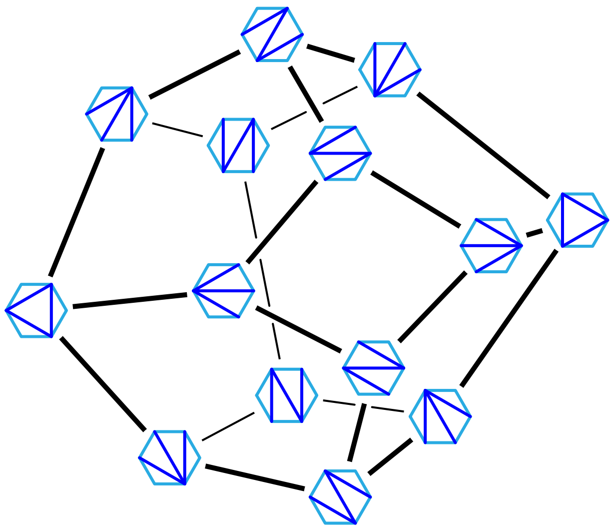
Tamari ('51) — Stasheff ('63) — Haimann ('84) — Lee ('89) —

... — Gel'fand-Kapranov-Zelevinski ('94) — ... — Chapoton-Fomin-Zelevinsky ('02) — ... — Loday ('04) — ...

— Ceballos-Santos-Ziegler ('11)

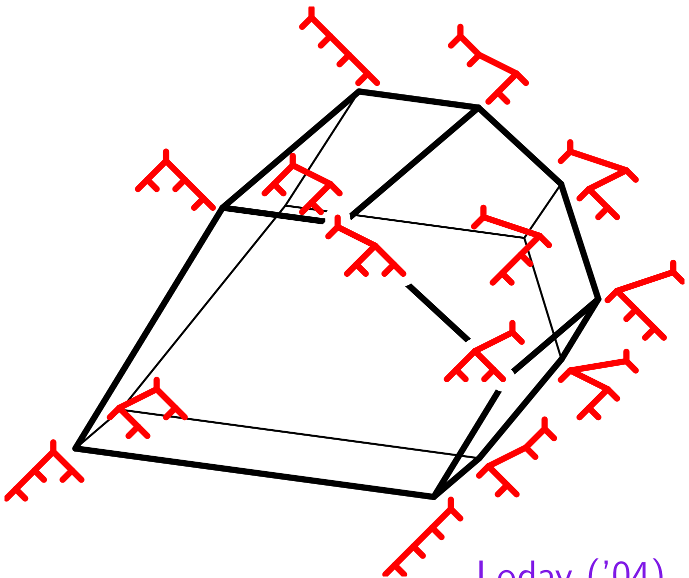
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

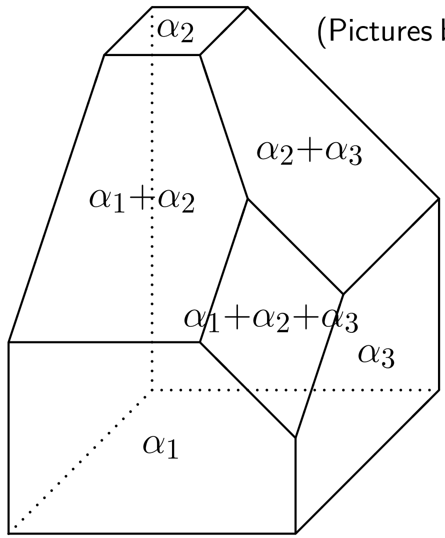
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

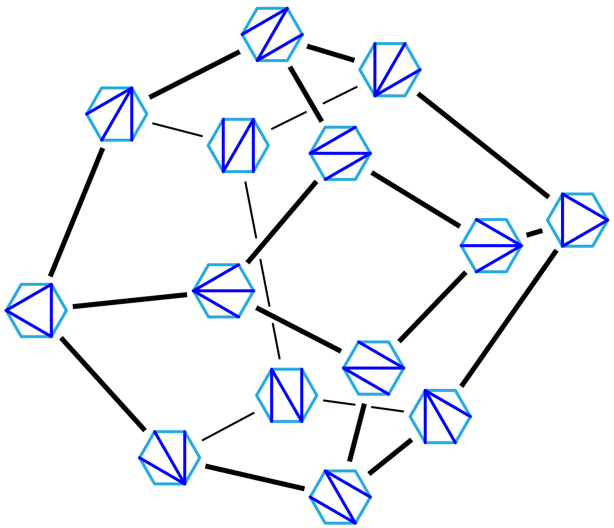
(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



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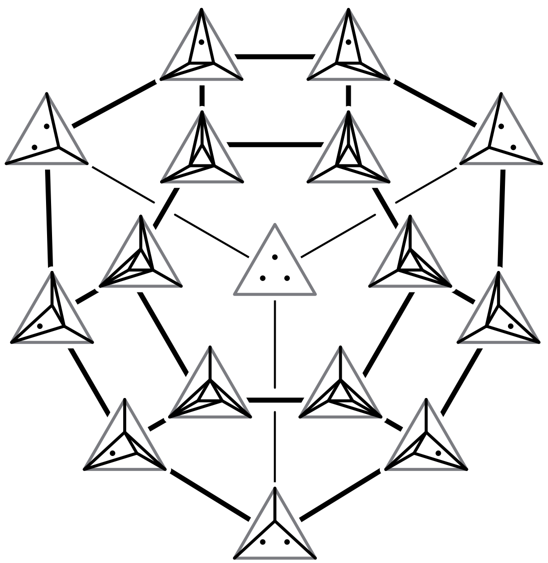
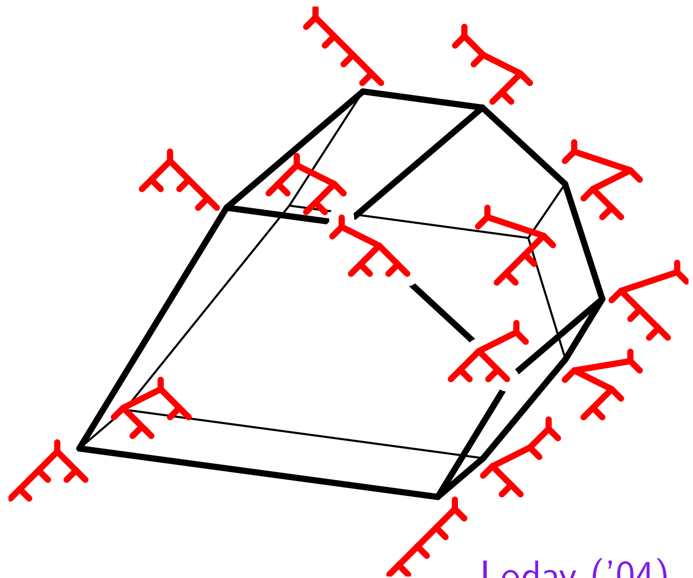


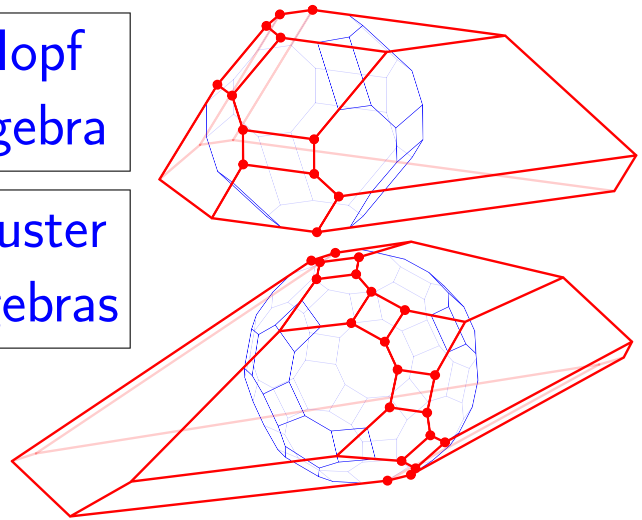
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

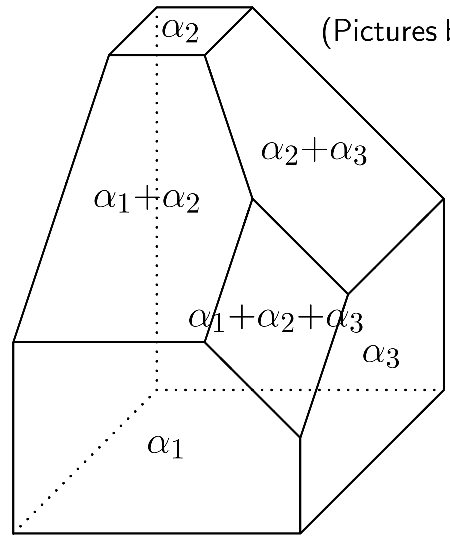
Hopf algebra

Cluster algebras



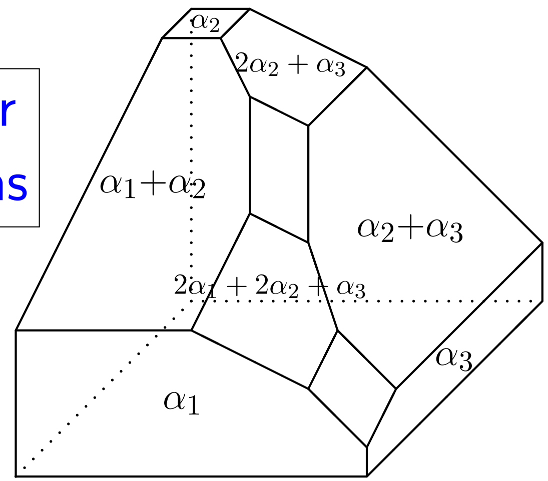
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras



GRAPH ASSOCIAHEDRA

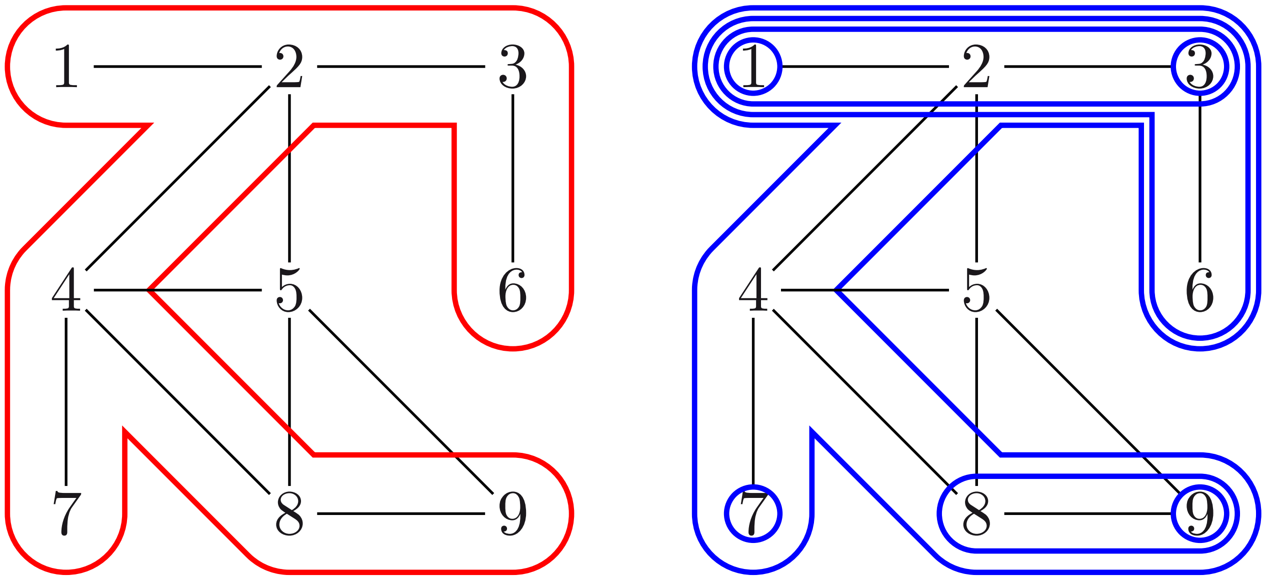
NESTED COMPLEX AND GRAPH ASSOCIAHEDRON

G graph on ground set V

Tube of G = connected induced subgraph of G

Compatible tubes = nested, or disjoint and non-adjacent

Tubing on G = collection of pairwise compatible tubes of G



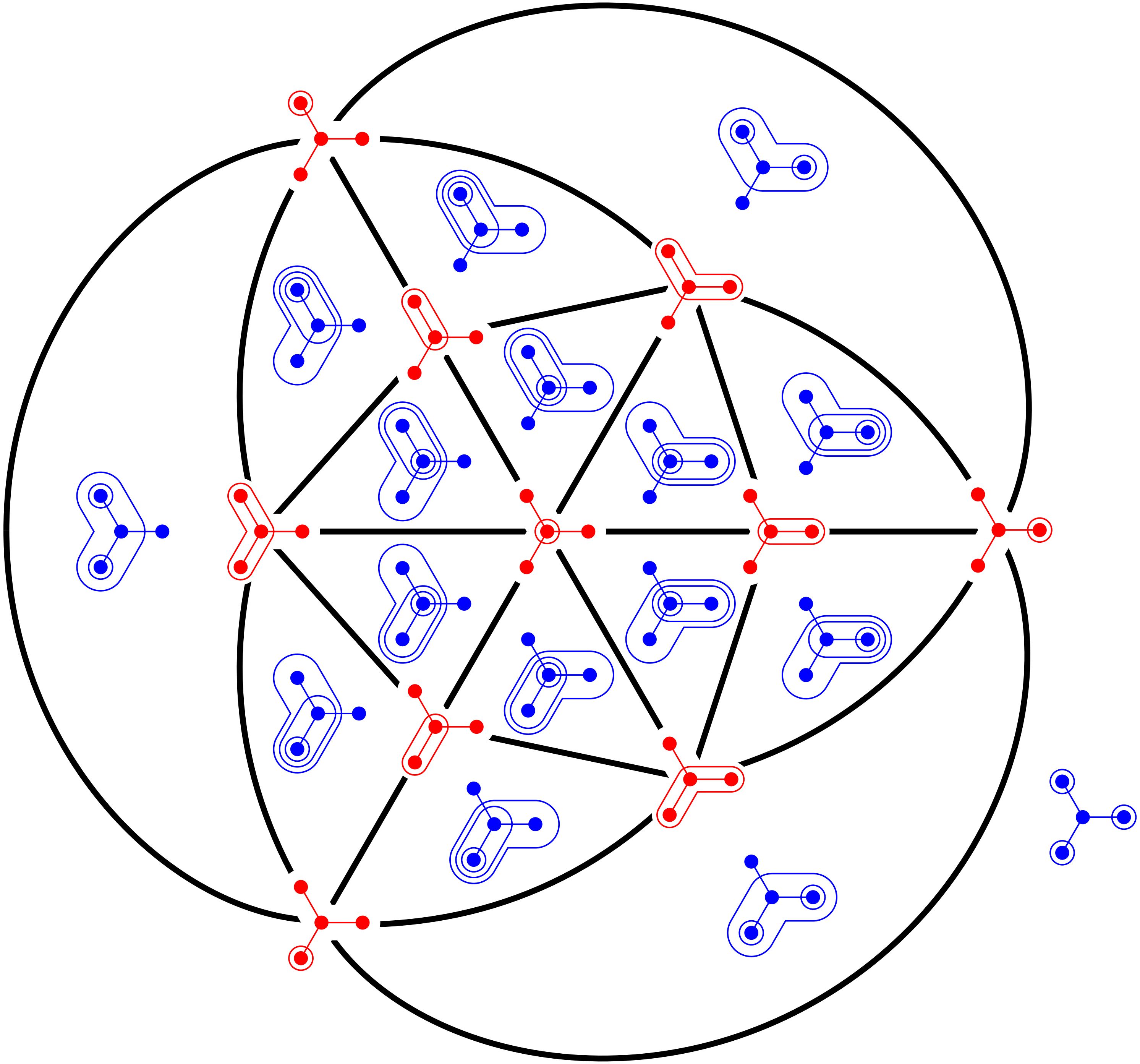
Nested complex $\mathcal{N}(\mathrm{G})$ = simplicial complex of tubings on G

= clique complex of the compatibility relation on tubes

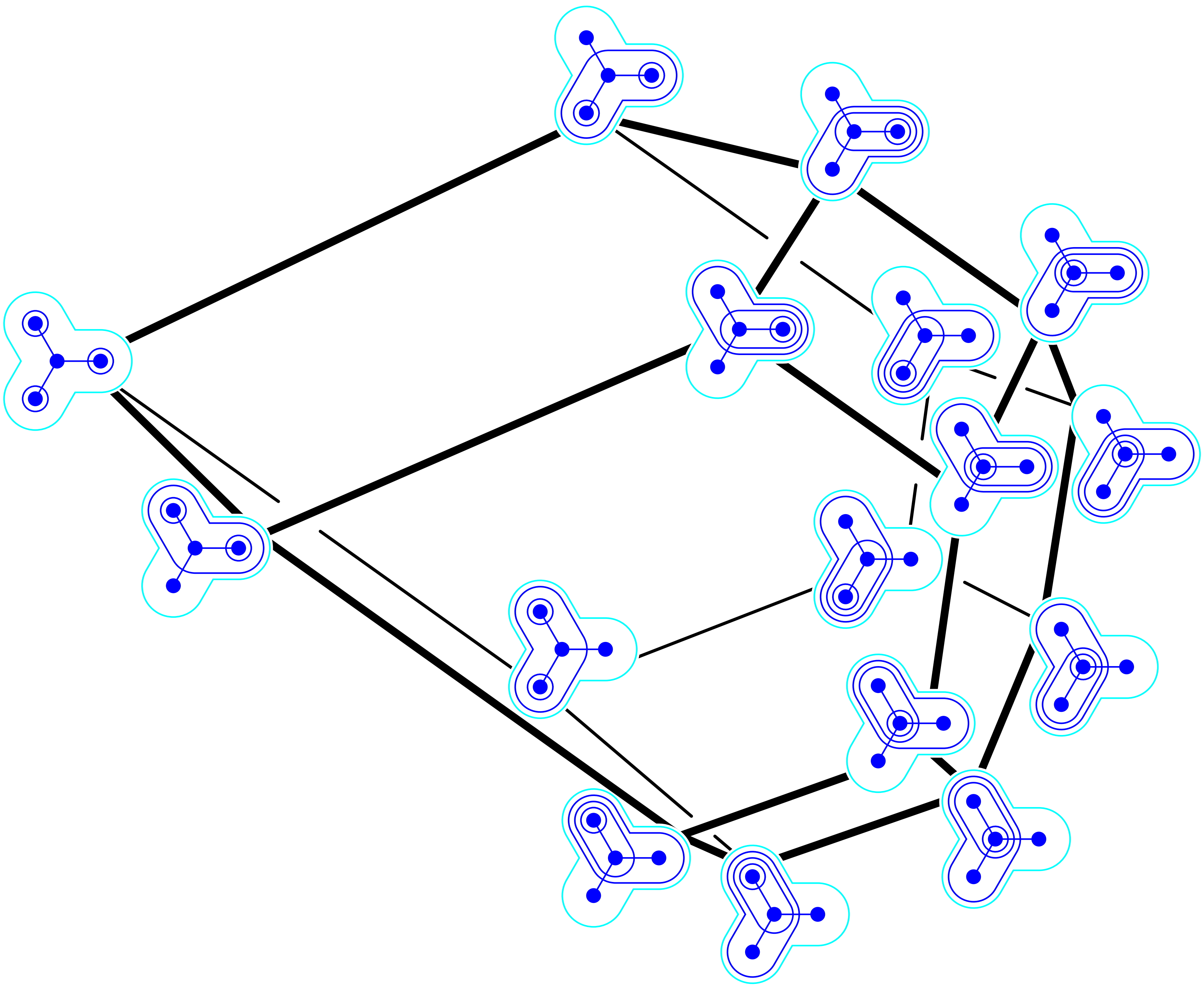
G-associahedron = polytopal realization of the nested complex on G

Carr-Devadoss, Coxeter complexes and graph associahedra ('06)

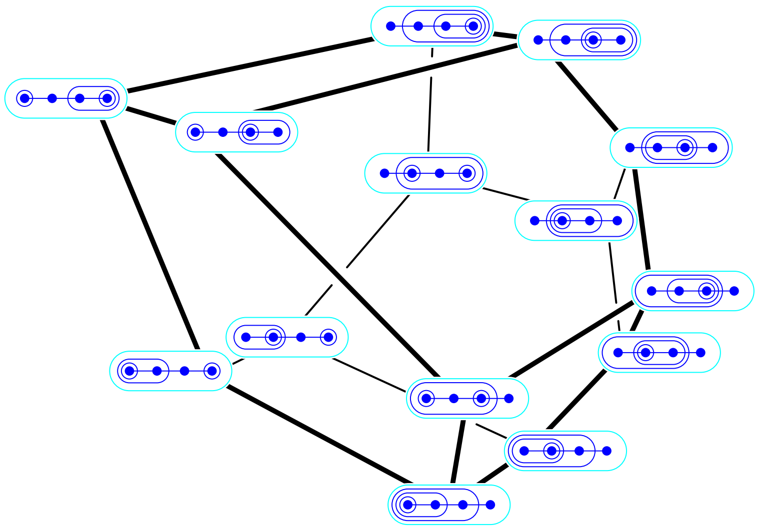
EXM: NESTED COMPLEX

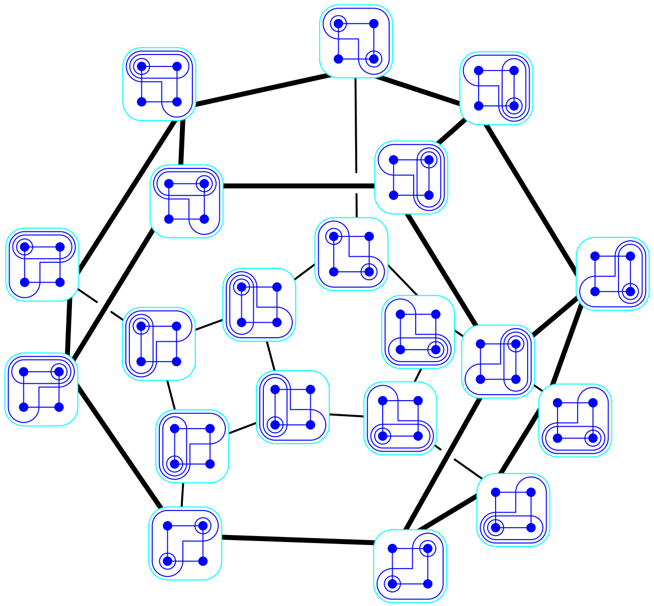


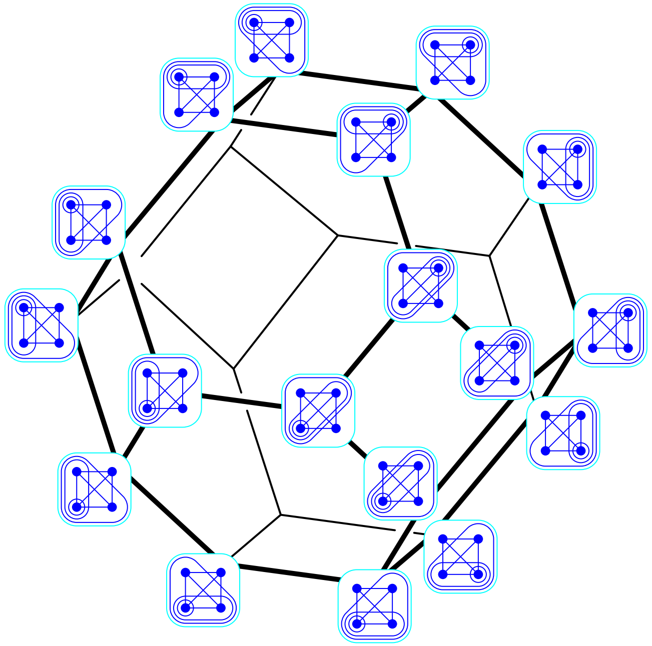
EXM: GRAPH ASSOCIAHEDRON



SPECIAL GRAPH ASSOCIAHEDRA



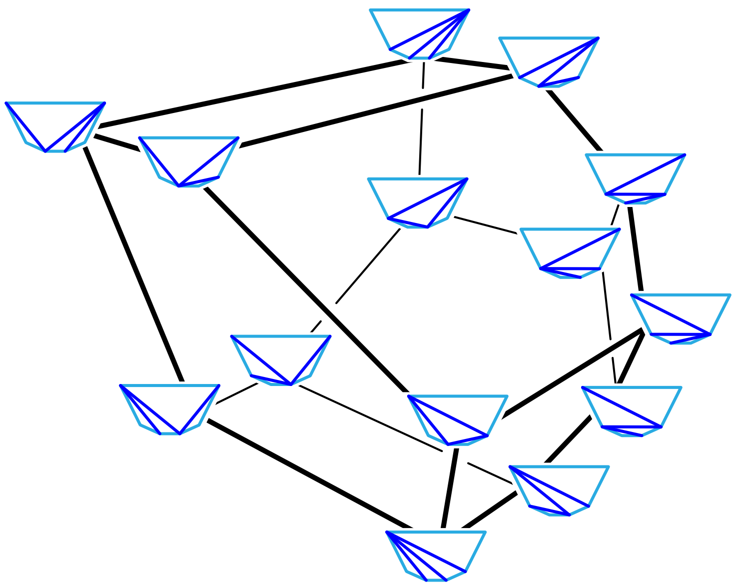


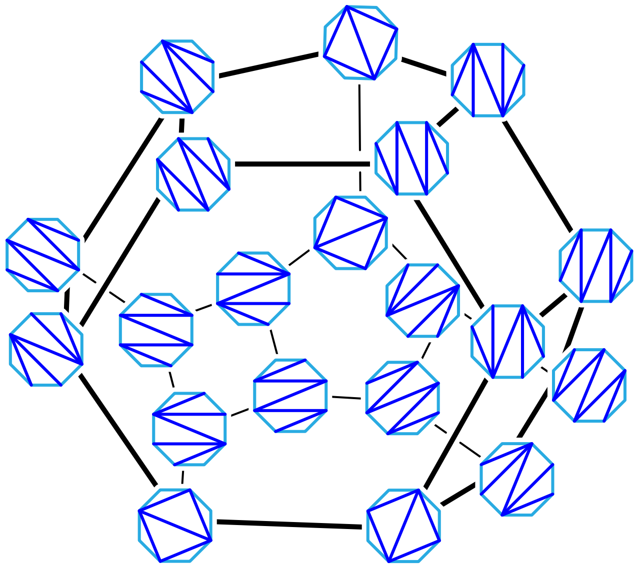


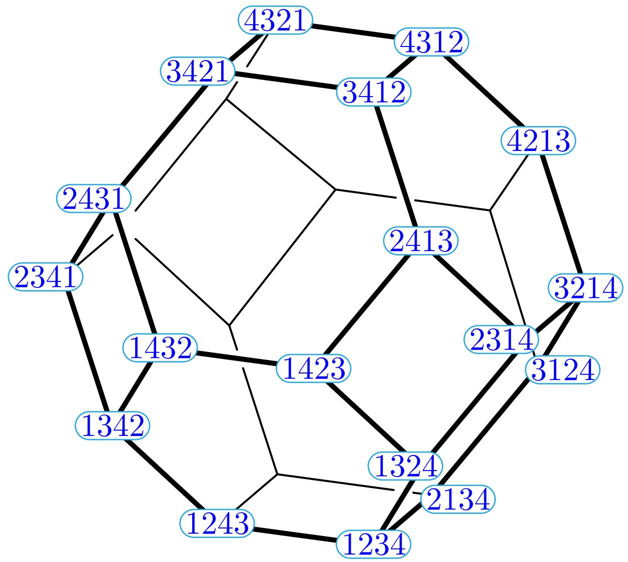
Path associahedron
= associahedron

Cycle associahedron
= cyclohedron

Complete graph associahedron
= permutahedron







LINEAR LAURENT PHENOMENON ALGEBRAS

Laurent Phenomenon Algebra = commut. ring gen. by cluster variables grouped in clusters

seed = pair $(\mathbf{x}, \mathbf{F})$ where

- $\mathbf{x} = \{x_1, \ldots, x_n\}$ cluster variables
- $\mathbf{F} = \{F_1, \ldots, F_n\}$ exchange polynomials

seed mutation = $(\mathbf{x}, \mathbf{F}) \longmapsto \mu_i(\mathbf{x}, \mathbf{F}) = (\mathbf{x}', \mathbf{F}')$ where

- $x_i' = \hat{F}_i / x_i$ while $x_j' = x_j$ for $j \neq i$
- F_j' obtained from F_j by eliminating x_i

THM. Every cluster variable in a LP algebra is a Laurent polynomial in the cluster variables of any seed.

Lam-Pylyavskyy, Laurent Phenomenon Algebras ('12)

Connection to graph associahedra: Any (directed) graph G defines a linear LP algebra whose cluster complex contains the nested complex of G

Lam-Pylyavskyy, Linear Laurent Phenomenon Algebras ('12)

COMPATIBILITY FANS FOR GRAPHICAL NESTED COMPLEXES

Thibault Manneville & VP

arXiv:1501.07152

COMPATIBILITY FANS FOR ASSOCIAHEDRA

T° an initial triangulation

δ, δ' two internal diagonals

compatibility degree between δ and δ'

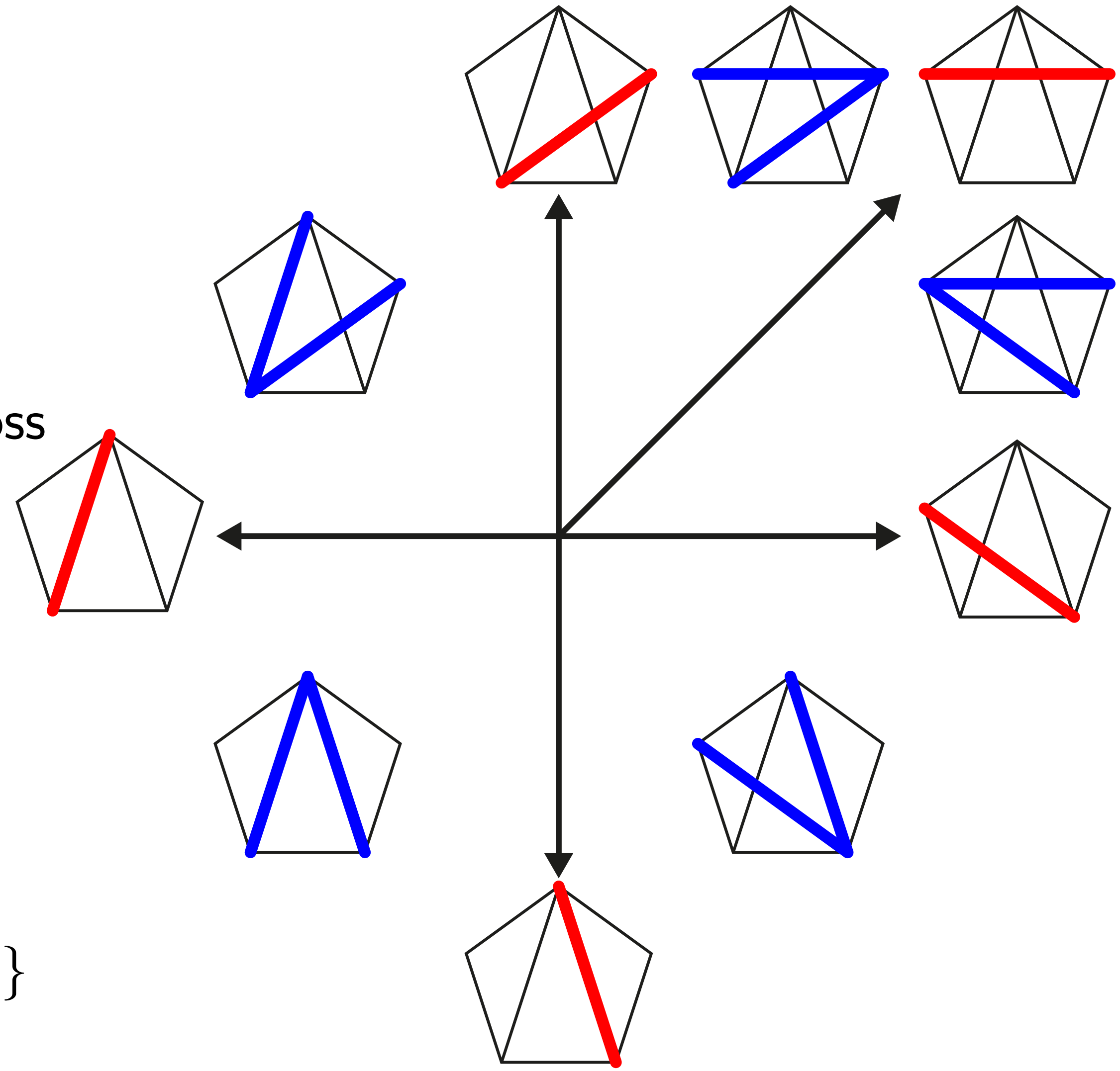
$$(\delta \parallel \delta') = \begin{cases} -1 & \text{if } \delta = \delta' \\ 0 & \text{if } \delta \text{ and } \delta' \text{ do not cross} \\ 1 & \text{if } \delta \text{ and } \delta' \text{ cross} \end{cases}$$

compatibility vector of δ wrt T°:

$$\mathbf{d}(\mathrm{T}^\circ, \delta) = \big[(\delta^\circ \parallel \delta)\big]_{\delta^\circ \in \mathrm{T}^\circ}$$

compatibility fan wrt T°

$$\mathcal{D}(\mathrm{T}^\circ) = \{\mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D} \text{ dissection}\}$$



Fomin-Zelevinsky, Y-Systems and generalized associahedra ('03)

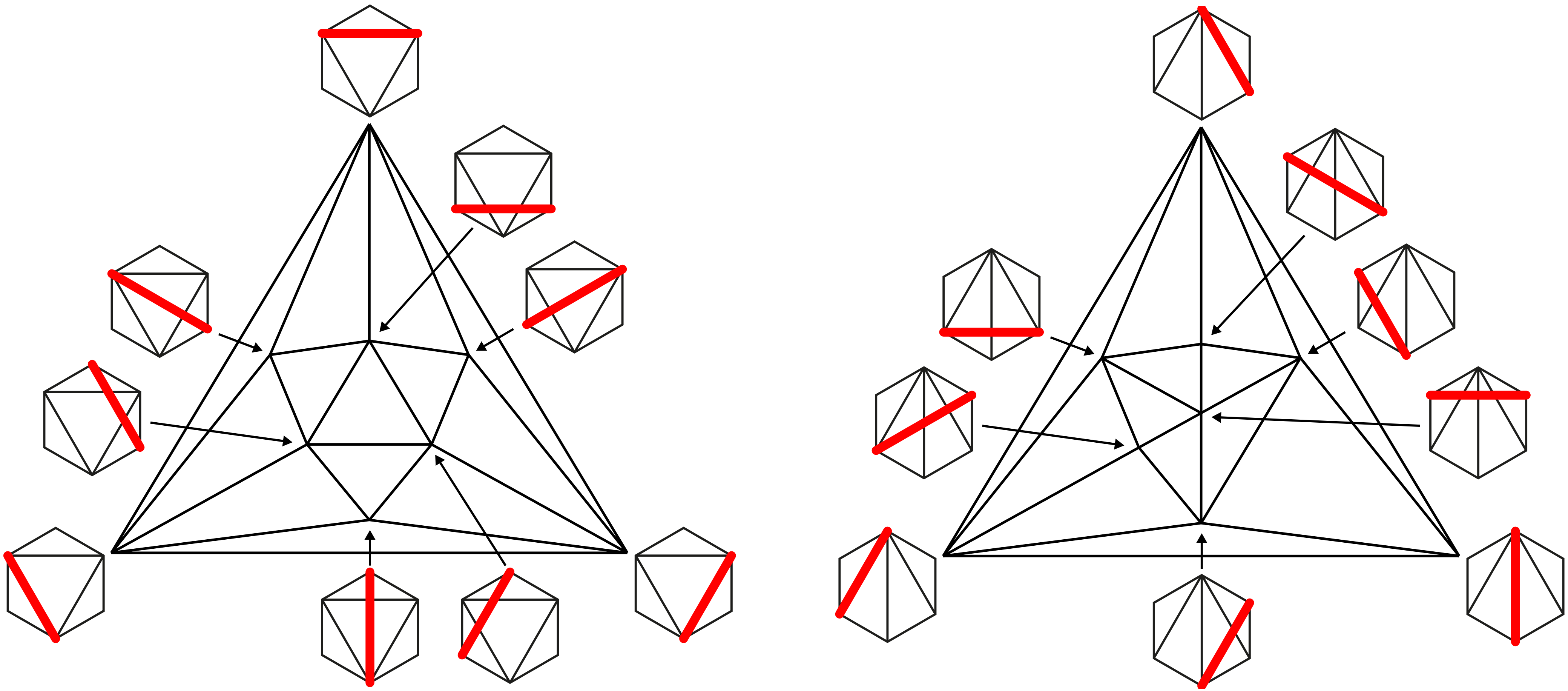
Fomin-Zelevinsky, Cluster algebras II: Finite type classification ('03)

Chapoton-Fomin-Zelevinsky, Polytopal realizations of generalized associahedra ('02)

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)

COMPATIBILITY FANS FOR ASSOCIAHEDRA

Different initial triangulations T° yield different realizations



THM. For any initial triangulation T°, the cones $\{\mathbb{R}_{\geq 0}\,\mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D} \text{ dissection}\}$ form a complete simplicial fan. Moreover, this fan is always polytopal.

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)

COMPATIBILITY FANS FOR GRAPHICAL NESTED COMPLEXES

T° an initial maximal tubing on G

t, t' two tubes of G

compatibility degree between t and t'

$$(\mathrm{t} \parallel \mathrm{t}') = \begin{cases} -1 & \text{if } \mathrm{t} = \mathrm{t}' \\ 0 & \text{if } \mathrm{t}, \mathrm{t}' \text{ are compatible} \\ |\{\text{neighbors of } \mathrm{t} \text{ in } \mathrm{t}' \smallsetminus \mathrm{t}\}| & \text{otherwise} \end{cases}$$

compatibility vector of t wrt T°:

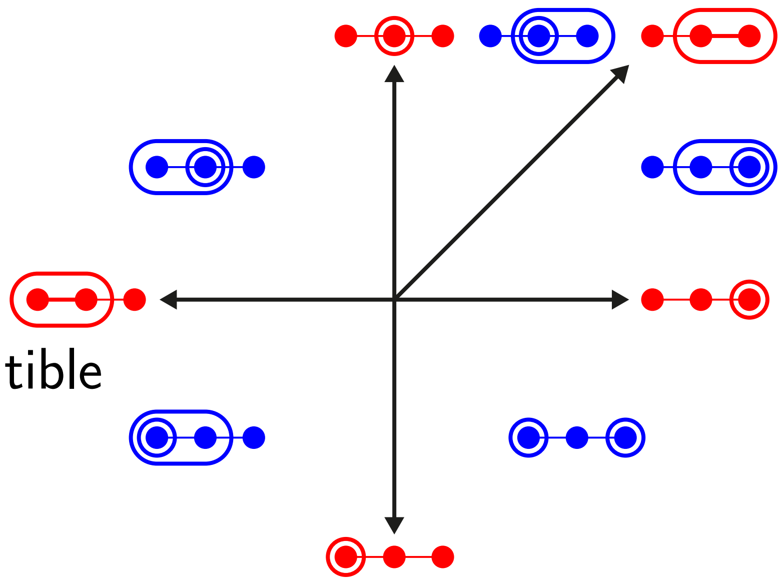
$$\mathbf{d}(\mathrm{T}^\circ, \mathrm{t}) = \left[(\mathrm{t}^\circ \parallel \mathrm{t})\right]_{\mathrm{t}^\circ \in \mathrm{T}^\circ}$$

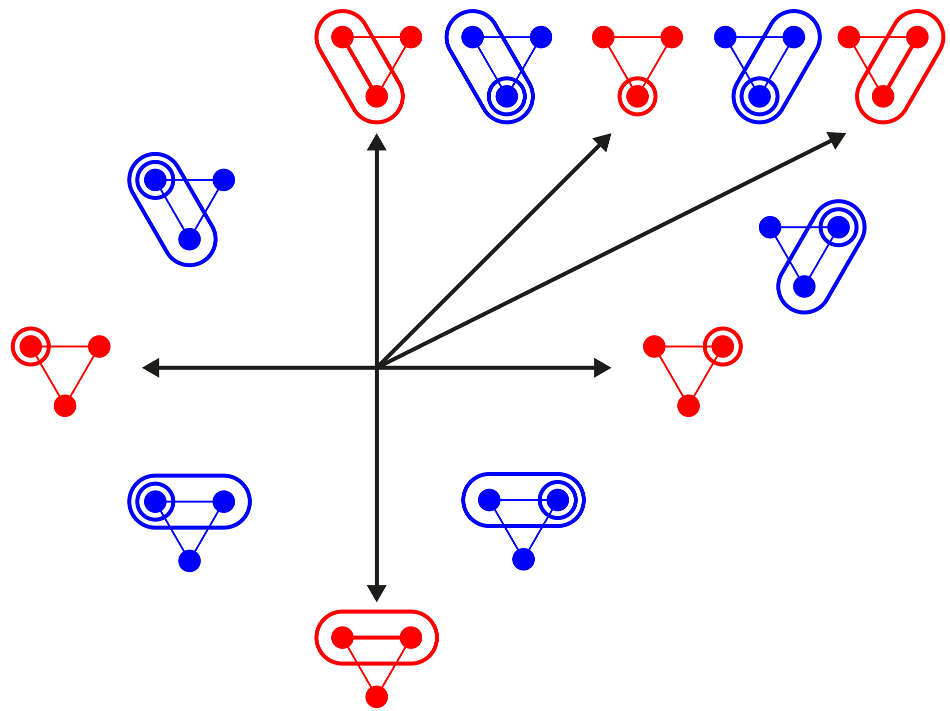
THM. For any initial maximal tubing T° on G, the collection of cones

$$\mathcal{D}(\mathrm{G}, \mathrm{T}^\circ) = \{\mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}^\circ, \mathrm{T}) \mid \mathrm{T} \text{ tubing on } \mathrm{G}\}$$

forms a complete simplicial fan, called compatibility fan of G.

Manneville-P., Compatibility fans for graphical nested complexes



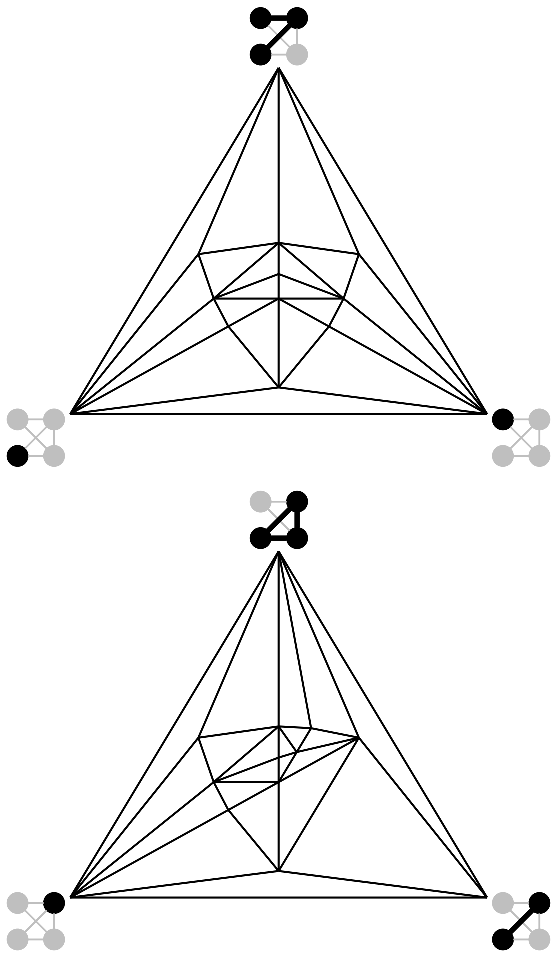


GRAPH CATALAN MANY SIMPLICIAL FAN REALIZATIONS

THM. When none of the connected components of G is a spider,

$$\# \text{ linear isomorphism classes of compatibility fans of } \mathrm{G}$$
$$= \# \text{ orbits of maximal tubings on } \mathrm{G} \text{ under graph automorphisms of } \mathrm{G}.$$

Manneville-P., Compatibility fans for graphical nested complexes ('15)



POLYTOPALITY?

QU. Are all compatibility fans polytopal?

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Polytopality of a complete simplicial fan $\iff$ Feasibility of a linear program

Exm: We check that the compatibility fan on the complete graph K_7 is polytopal by solving a linear program on 126 variables and $17\,640$ inequalities

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Exm: We check that the compatibility fan on the complete graph K_7 is polytopal by solving a linear program on 126 variables and $17\,640$ inequalities

$\implies$ All compatibility fans on complete graphs of ≤ 7 vertices are polytopal...
$\implies$ All compatibility fans on graphs of ≤ 4 vertices are polytopal...

POLYTOPALITY?

QU. Are all compatibility fans polytopal?

Polytopality of a complete simplicial fan $\iff$ Feasibility of a linear program

Exm: We check that the compatibility fan on the complete graph K_7 is polytopal by solving a linear program on 126 variables and $17\,640$ inequalities

$\Longrightarrow$ All compatibility fans on complete graphs of ≤ 7 vertices are polytopal...
$\Longrightarrow$ All compatibility fans on graphs of ≤ 4 vertices are polytopal...

To go further, we need to understand better the linear dependences between the compatibility vectors of the tubes involved in a flip

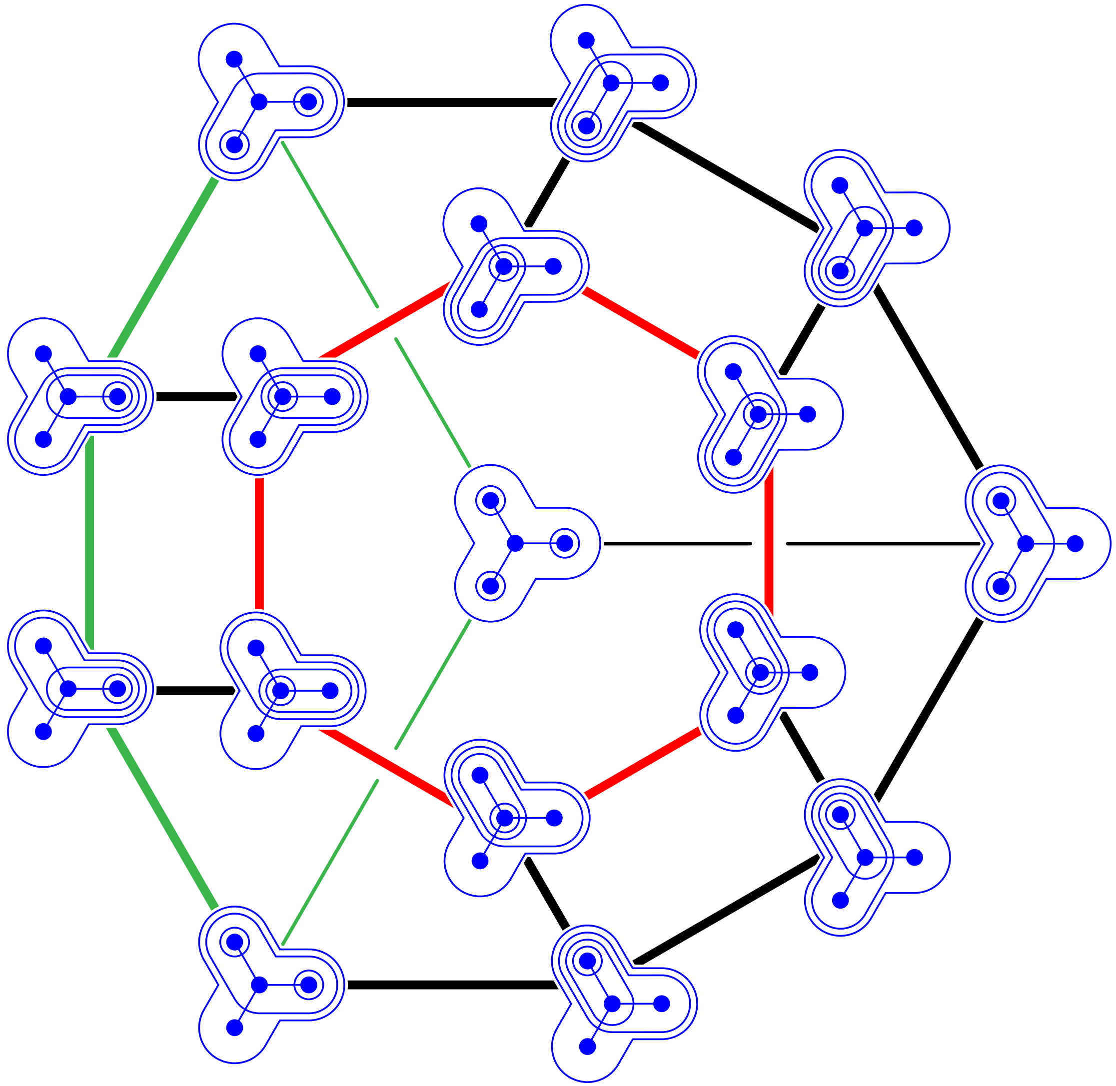
THM. All compatibility fans on the paths and cycles are polytopal

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)
Manneville-P., Compatibility fans for graphical nested complexes ('15)

POLYTOPALITY?

QU. Are all compatibility fans polytopal?

Remarkable realizations of the stellohedra



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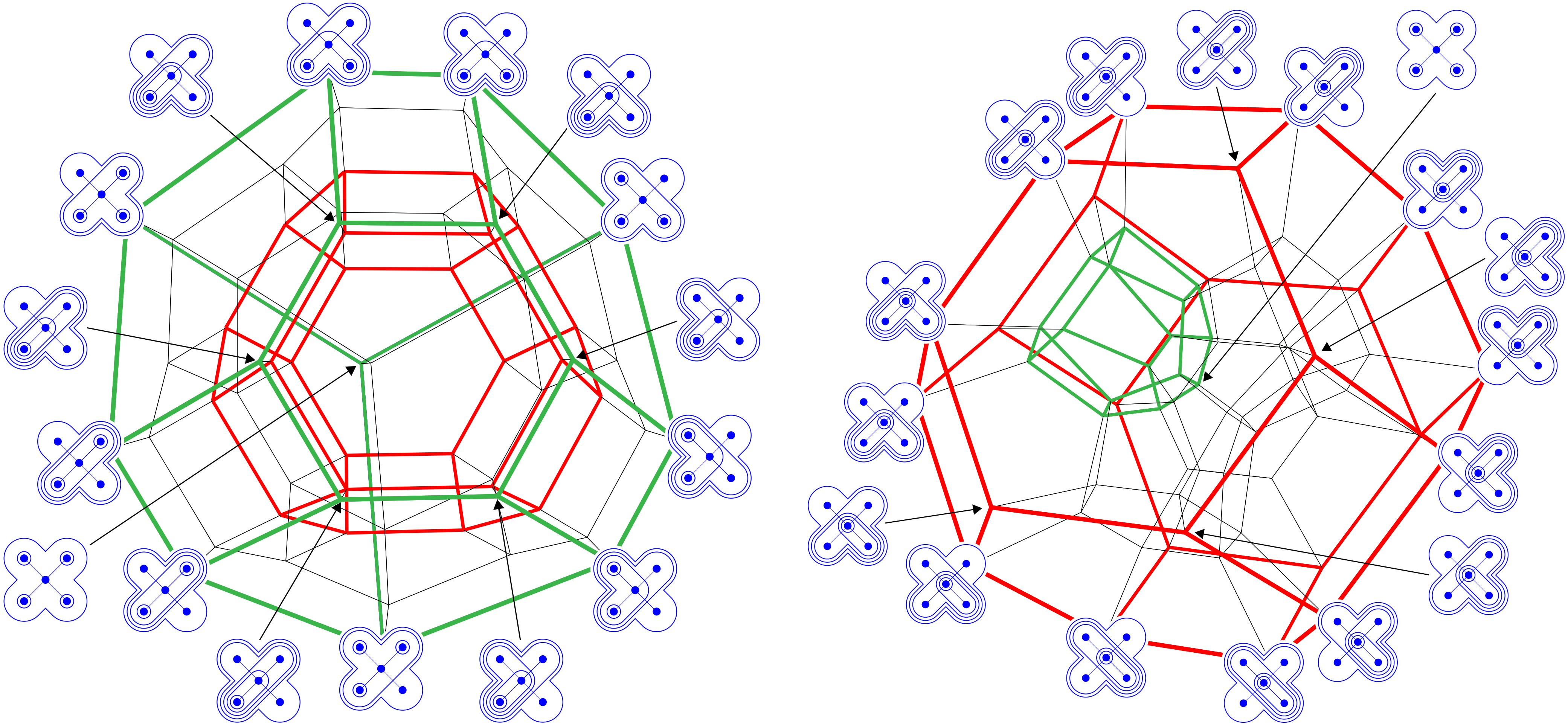


Convex hull of the orbits under coordinate permutations of the set $\left\{\sum_{i>k} i\,\mathbf{e}_i \;\middle|\; 0 \le k \le n\right\}$

SIGNED TREE ASSOCIAHEDRA

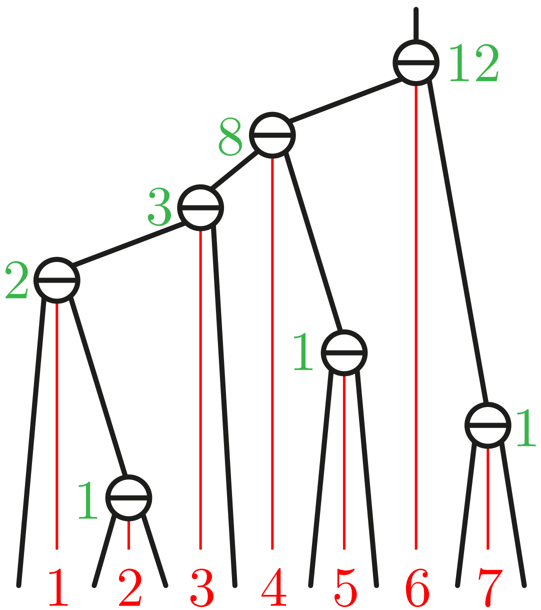
arXiv:1309.5222

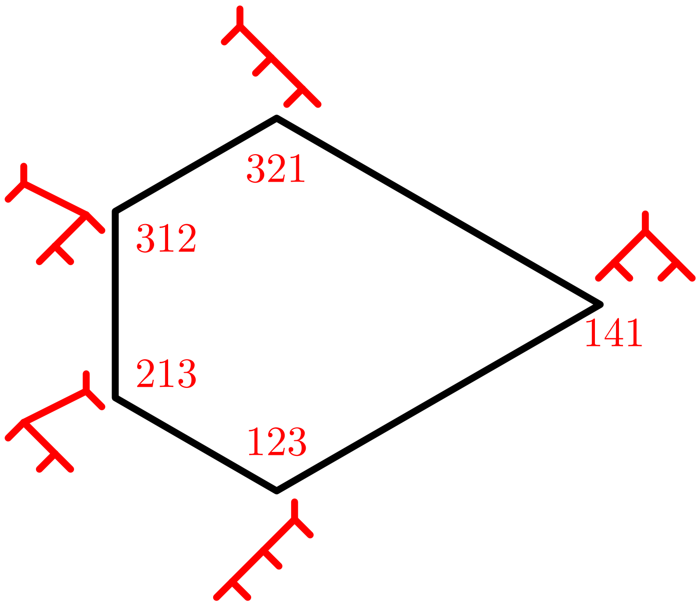
LODAY'S ASSOCIAHEDRON

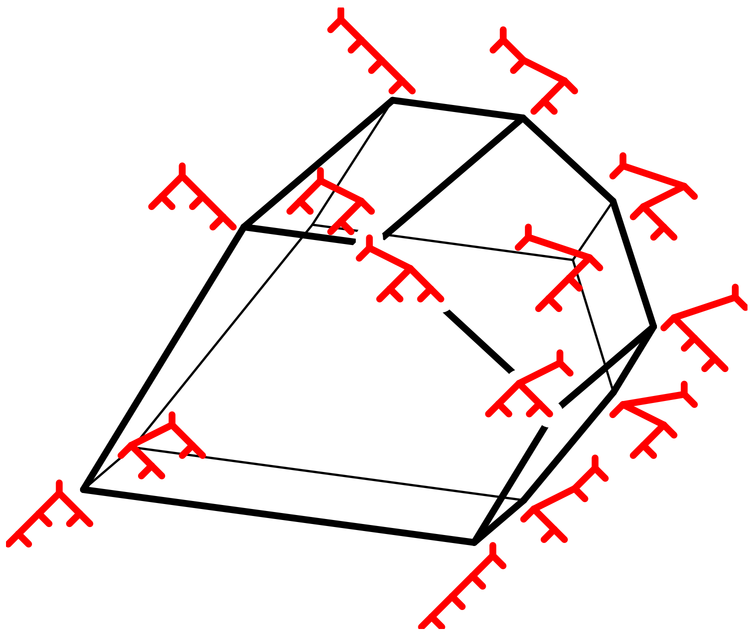
$$\mathsf{Asso}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1 \le i \le j \le n+1} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i)\cdot r(\mathrm{T},i)\big]_{i\in[n+1]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \le k \le j} x_i \ge \binom{j-i+2}{2} \right\}$$

Loday, Realization of the Stasheff polytope ('04)







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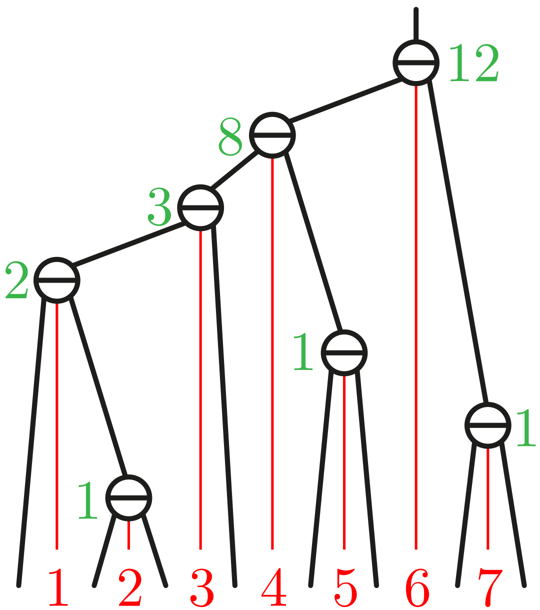
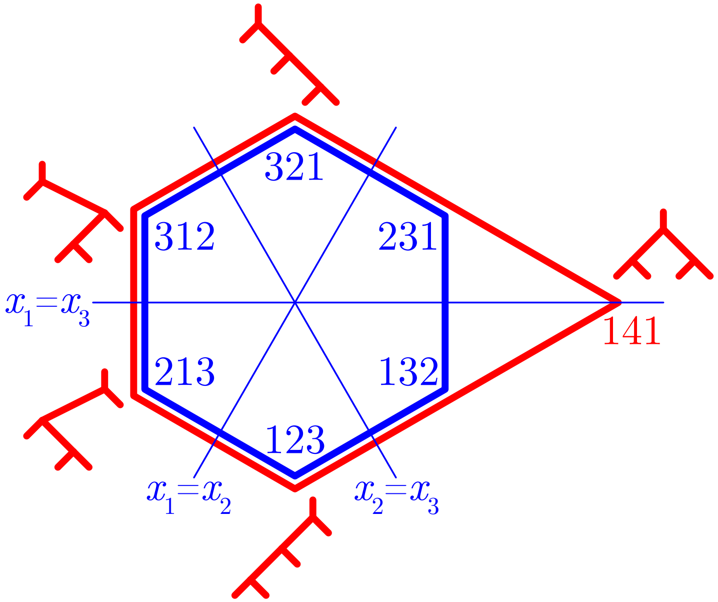
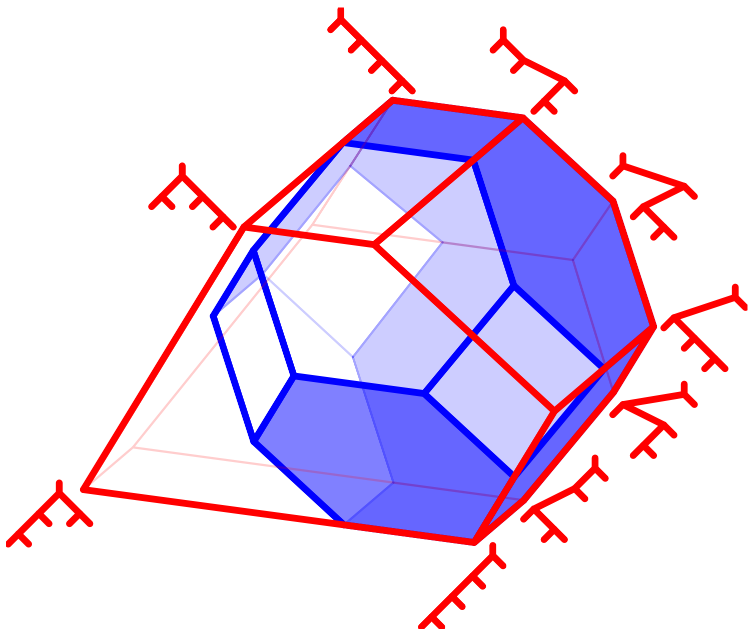




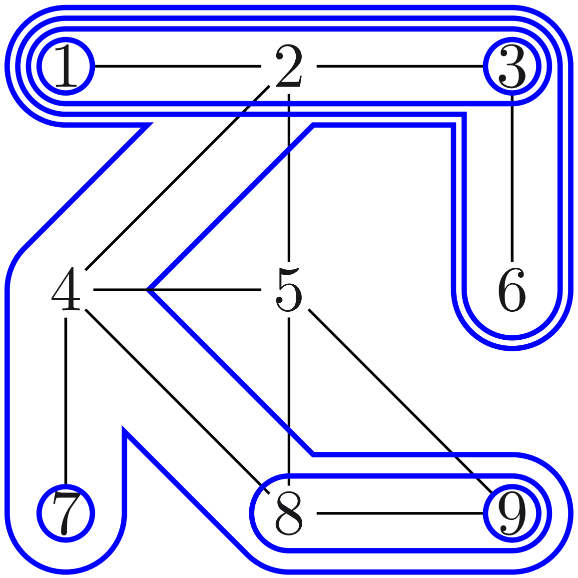


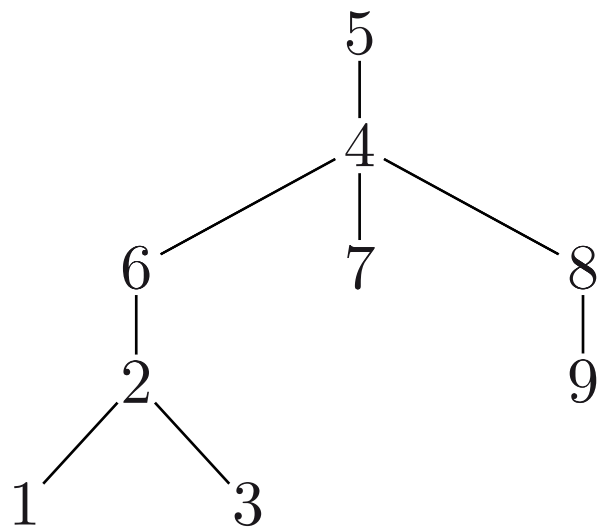
- Asso(n) obtained by deleting inequalities in the facet description of the permutahedron
- normal cone of $\mathbf{L}(\mathrm{T})$ in $\mathsf{Asso}(n) = \{\mathbf{x}\in\mathbb{H} \mid x_i < x_j \text{ for all } i \to j \text{ in } \mathrm{T}\}$

  $= \bigcup_{\sigma\in\mathcal{L}(\mathrm{T})}$ normal cone of σ in $\mathsf{Perm}(n)$

SPINES

spine of a tubing T = Hasse diagram of the inclusion poset of T





tube t of the tubing T $\longmapsto$ node $\mathrm{s}(\mathrm{t})$ of the spine S labeled by $\mathrm{t} \smallsetminus \bigcup \{\mathrm{t}' \mid \mathrm{t}' \in \mathrm{T}, \mathrm{t}' \subsetneq \mathrm{t}\}$

tube $\mathrm{t}(\mathrm{s}) \coloneqq \bigcup \{\mathrm{s}' \mid \mathrm{s}' \le \mathrm{s} \text{ in } \mathrm{S}\}$ of the tubing T $\longleftarrow$ node s of the spine S

S spine on G $\iff$ for each node s of S with children $\mathrm{s}_1 \dots \mathrm{s}_k$, the tubes $\mathrm{t}(\mathrm{s}_1) \dots \mathrm{t}(\mathrm{s}_k)$ lie in distinct connected components of $\mathrm{G}\big[\mathrm{t}(\mathrm{s}) \smallsetminus \mathrm{s}\big]$

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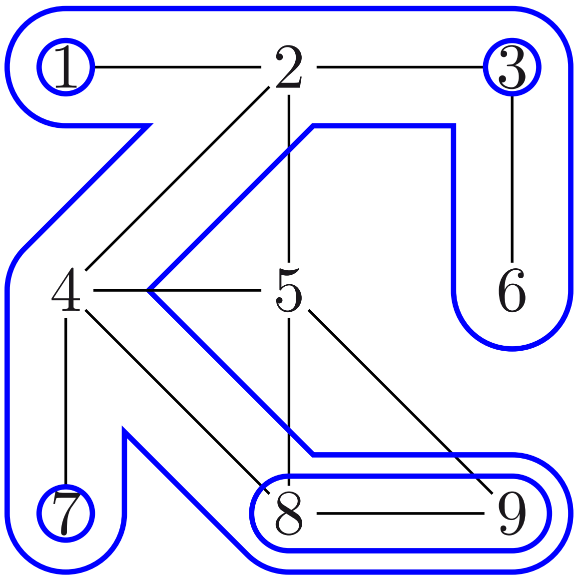

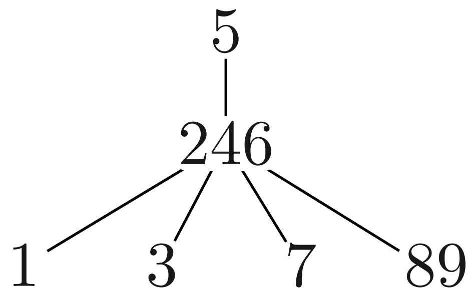

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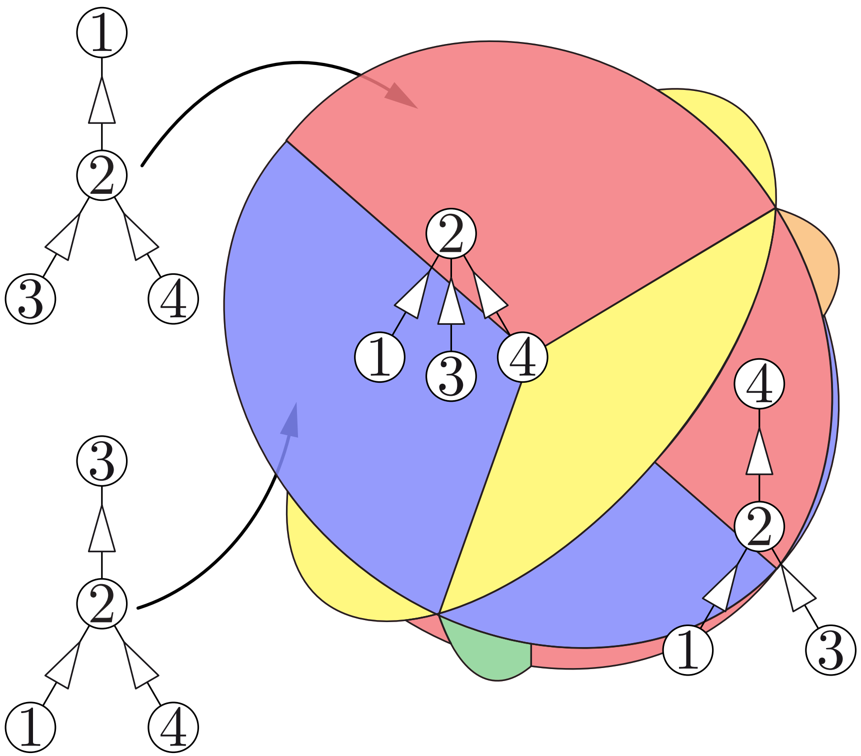
NESTED FANS AND GRAPH ASSOCIAHEDRA

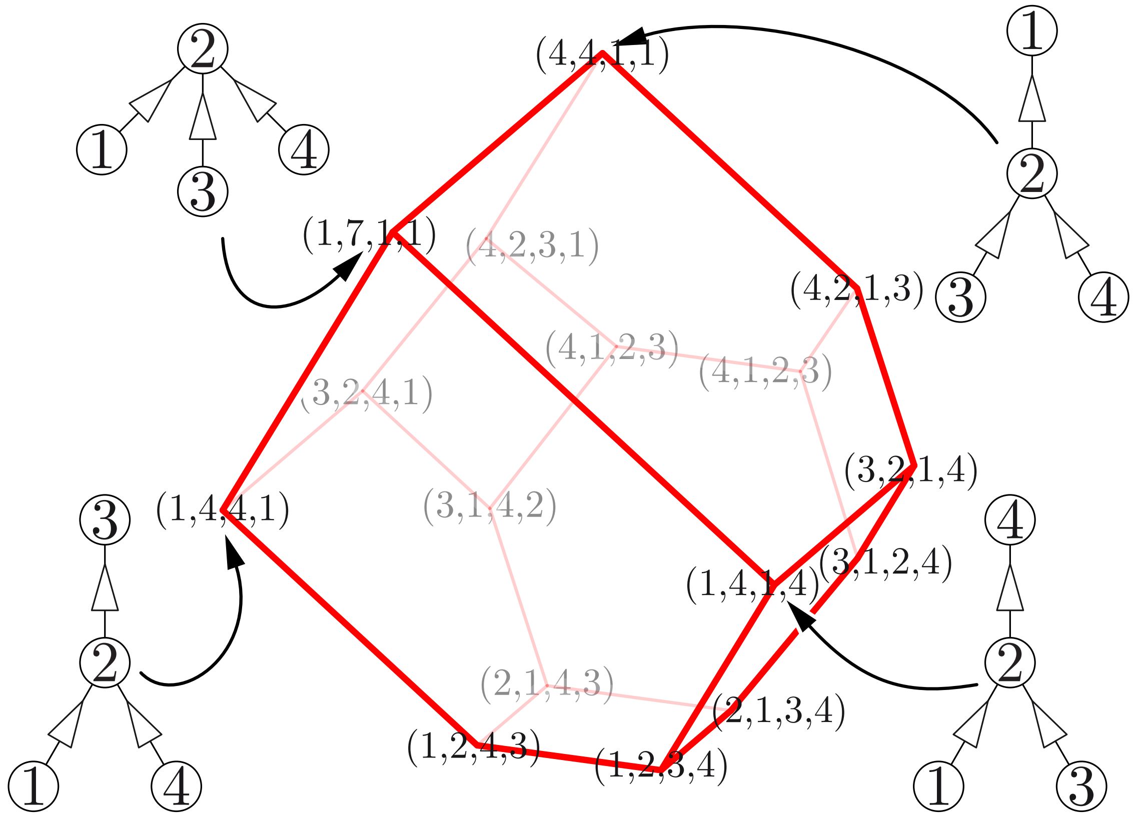
THM. The collection of cones $\big\{ \{\mathbf{x} \in \mathbb{H} \mid x_i < x_j \text{ for all } i \to j \text{ in } \mathrm{T}\} \;\big|\; \mathrm{T} \text{ tubing on } \mathrm{G} \big\}$ forms a complete simplicial fan, called the nested fan of G. This fan is always polytopal.

Carr-Devadoss, Coxeter complexes and graph associahedra ('06)

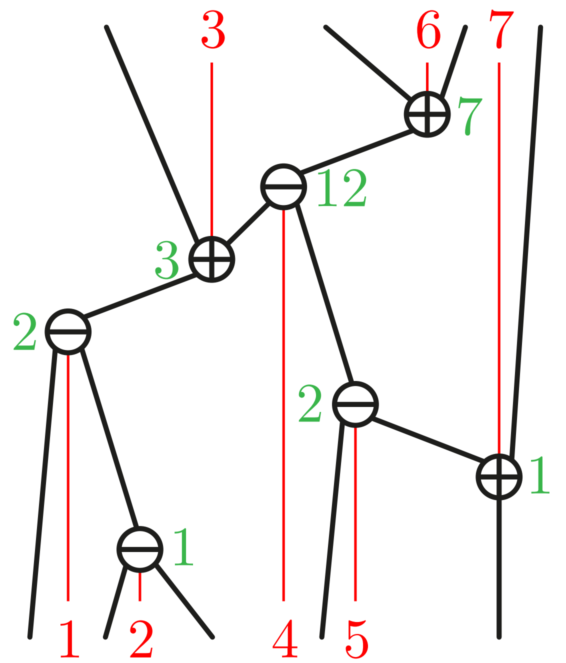
Postnikov, Permutohedra, associahedra, and beyond ('09)

Zelevinsky, Nested complexes and their polyhedral realizations ('06)





HOHLWEG-LANGE'S ASSOCIAHEDRA

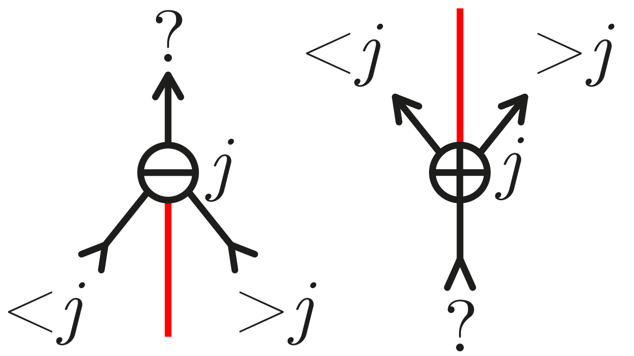


for an arbitrary signature $\varepsilon \in \pm^{n+1}$,

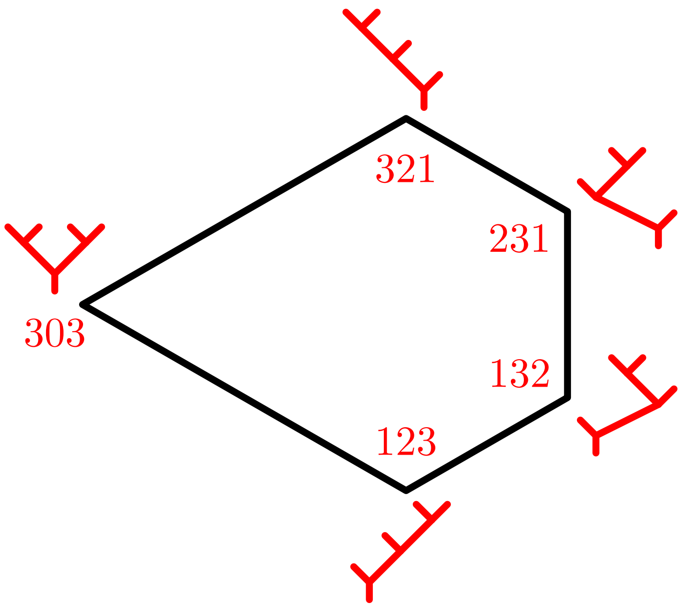
$$\mathsf{Asso}(\varepsilon) \;\coloneqq\; \mathrm{conv}\left\{\mathbf{HL}(\mathrm{T}) \;\middle|\; \mathrm{T}\ \varepsilon\text{-Cambrian tree}\right\}$$

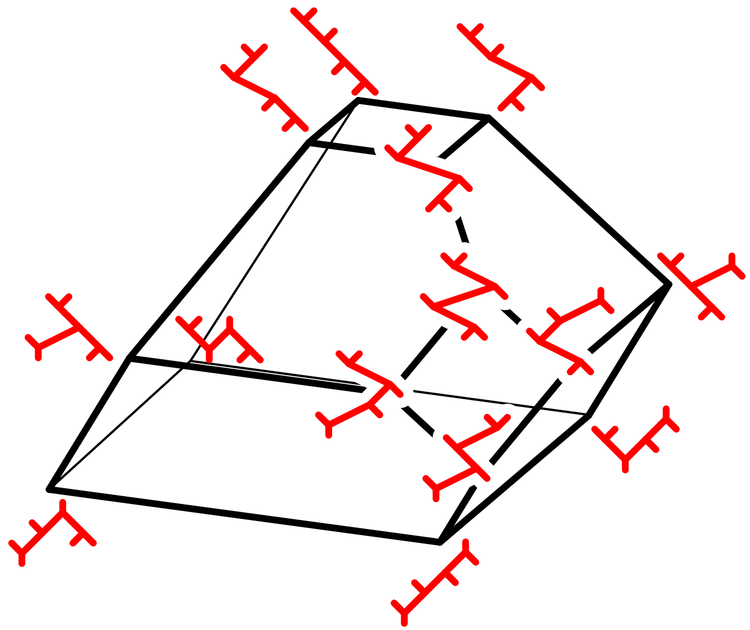
$$\text{with } \mathbf{HL}(\mathrm{T})_j \coloneqq \begin{cases} \ell(\mathrm{T},j)\cdot r(\mathrm{T},j) & \text{if } \varepsilon(j) = - \\ n+2-\ell(\mathrm{T},j)\cdot r(\mathrm{T},j) & \text{if } \varepsilon(j) = + \end{cases}$$

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)
Lange-P., *Using spines to revisit a construction of the associahedron* ('15)



Rule for Cambrian trees





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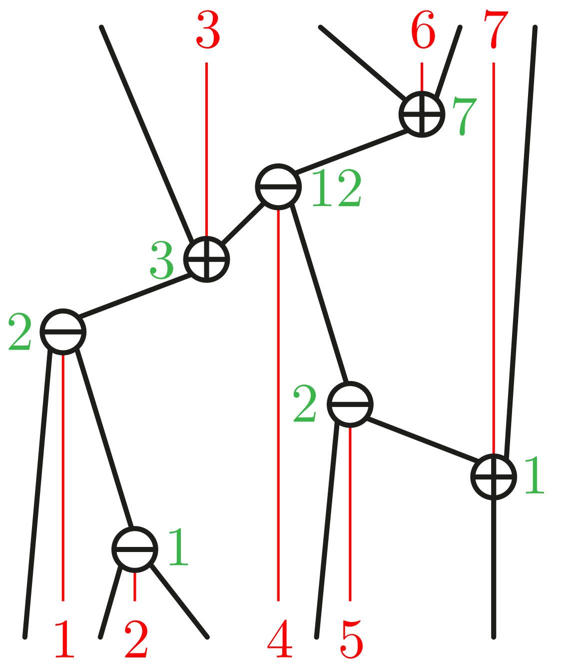

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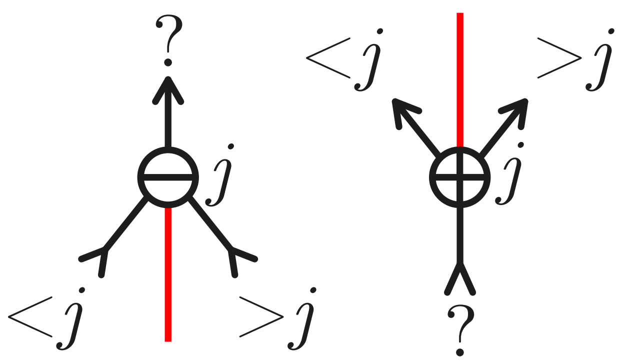

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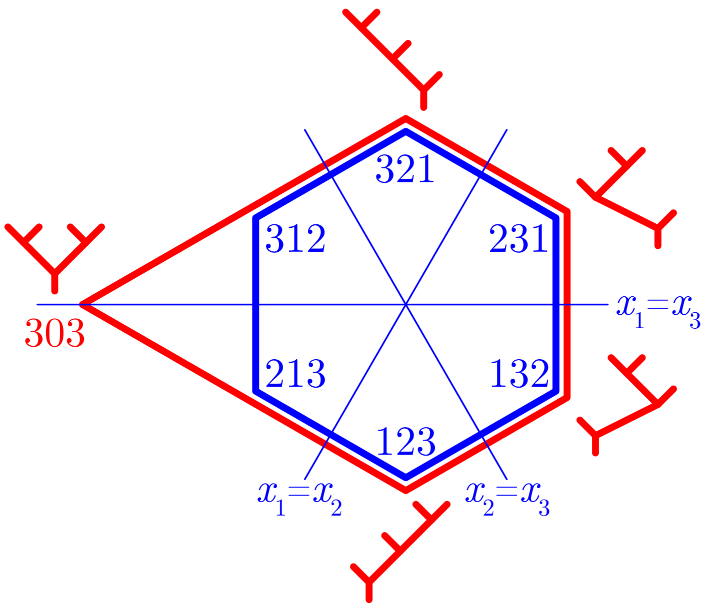

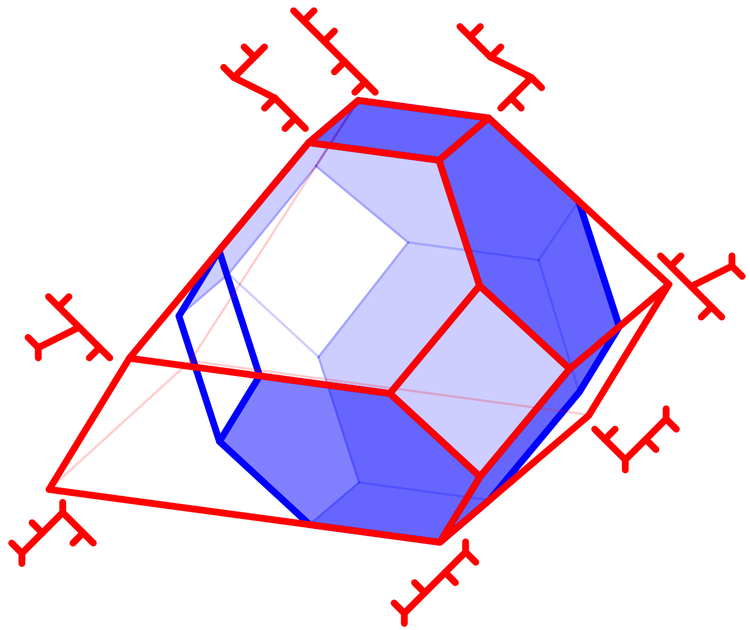

- Asso(n) obtained by deleting inequalities in the facet description of the permutahedron
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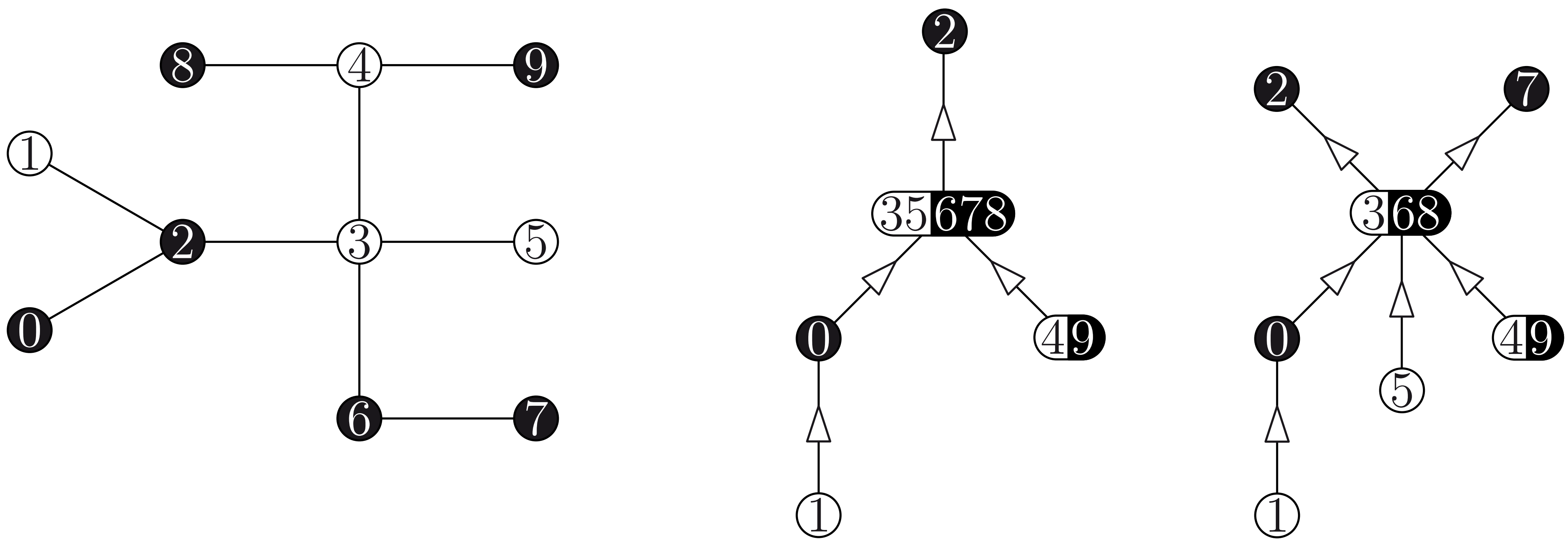
SIGNED SPINES ON SIGNED TREES

T tree on the signed ground set $\mathrm{V} = \mathrm{V}^- \sqcup \mathrm{V}^+$ (negative in white, positive in black)

Signed spine on T = directed and labeled tree S st

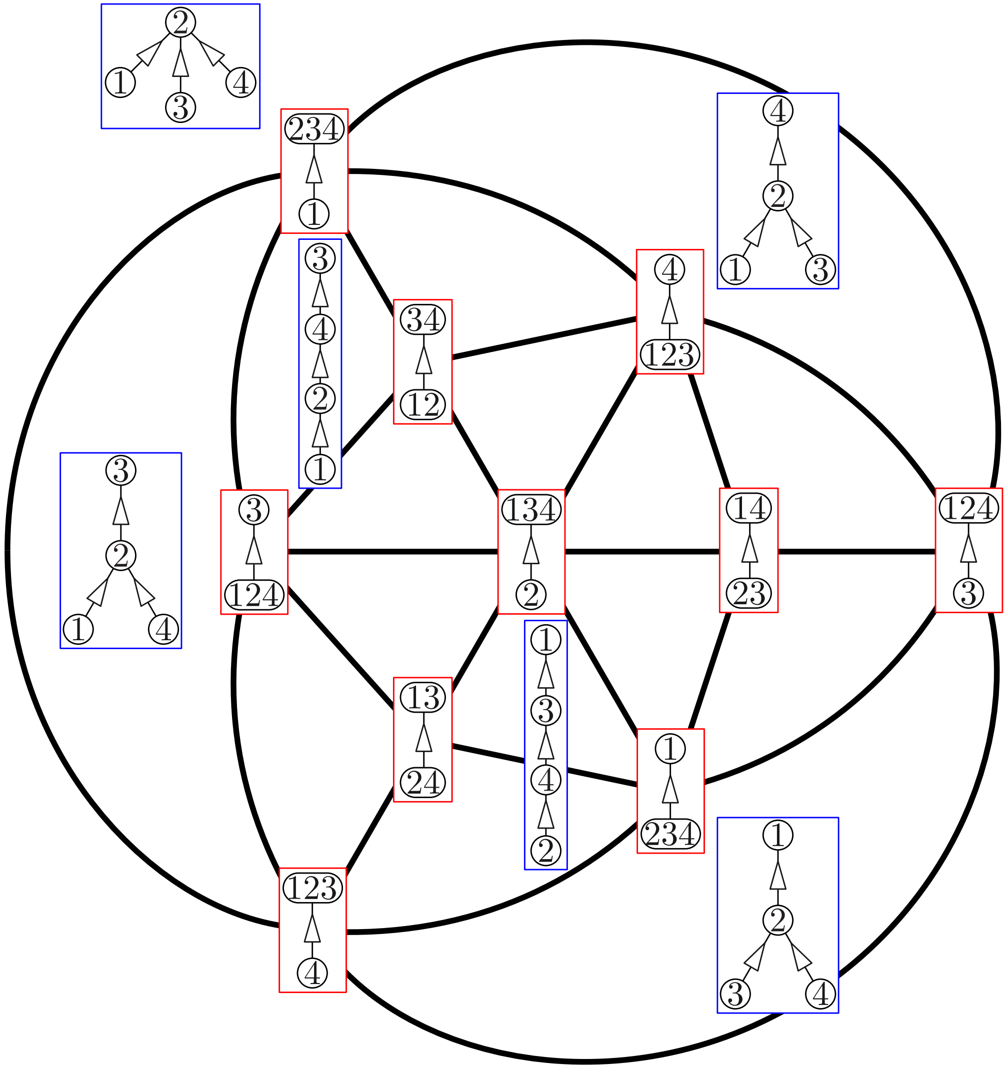
(i) the labels of the nodes of S form a partition of the signed ground set V

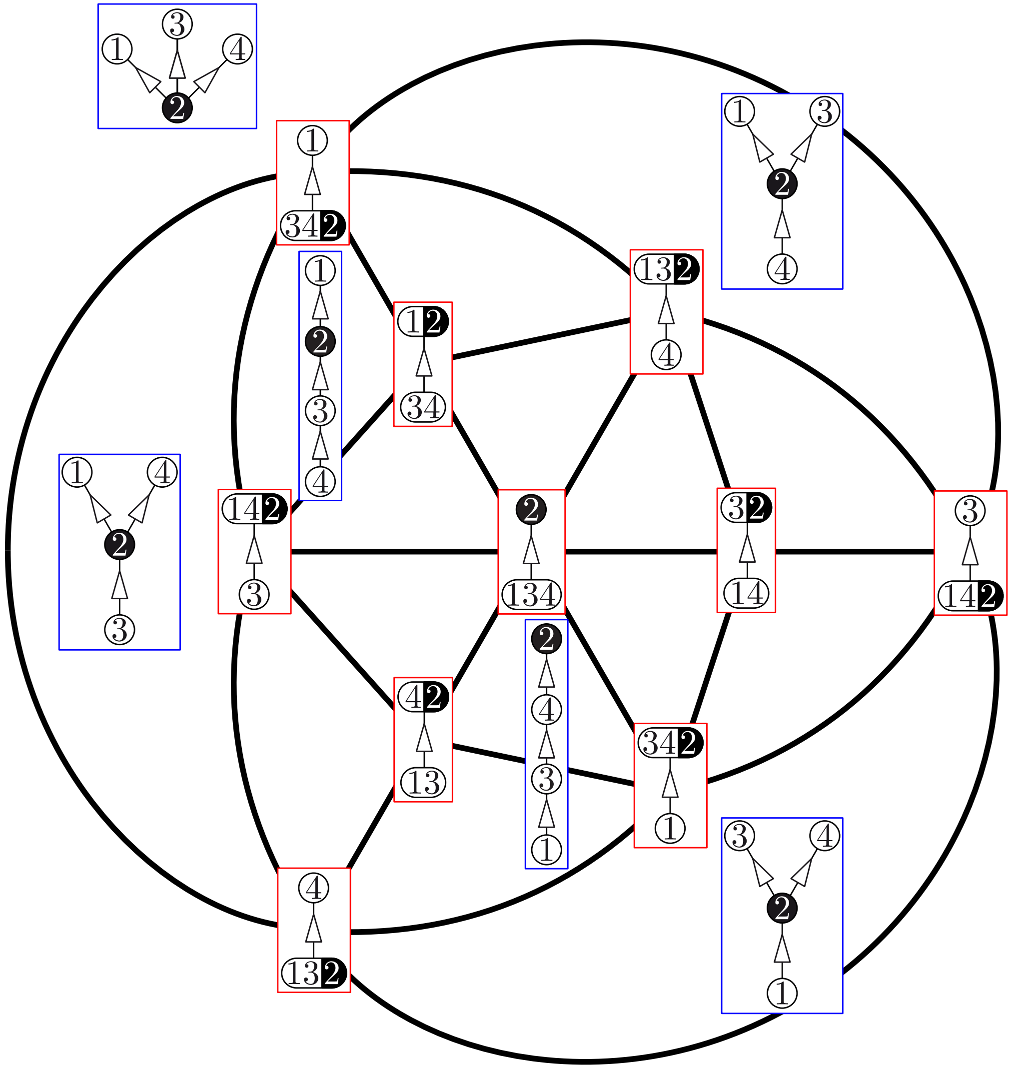
(ii) at a node of S labeled by $U = U^- \sqcup U^+$, the source label sets of the different incoming arcs are subsets of distinct connected components of $\mathrm{T} \smallsetminus U^-$, while the sink label sets of the different outgoing arcs are subsets of distinct connected components of $\mathrm{T} \smallsetminus U^+$



SPINE COMPLEX

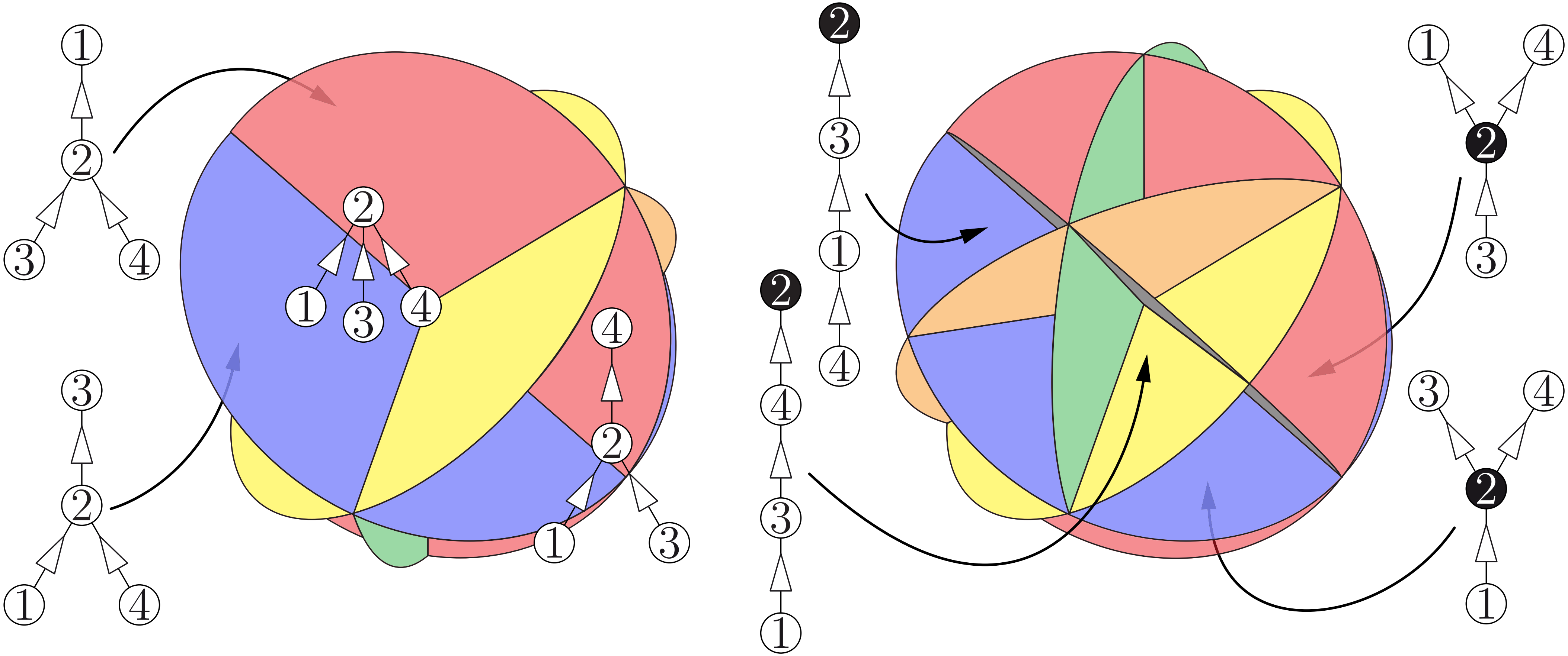
Signed spine complex $\mathcal{S}(\mathrm{T})$ = simplicial complex whose inclusion poset is isomorphic to the poset of edge contractions on the signed spines of T





SPINE FAN

For S spine on T, define $\mathrm{C}(\mathrm{S}) \coloneqq \{\mathbf{x} \in \mathbb{H} \mid x_u \leq x_v, \text{ for all arcs } u \to v \text{ in } \mathrm{S}\}$



THEO. The collection of cones $\mathcal{F}(\mathrm{T}) \coloneqq \{\mathrm{C}(\mathrm{S}) \mid \mathrm{S} \in \mathcal{S}(\mathrm{T})\}$ defines a complete simplicial fan on $\mathbb{H}$, which we call the spine fan

SIGNED TREE ASSOCIAHEDRA

THM. The spine fan $\mathcal{F}(\mathrm{T})$ is the normal fan of the signed tree associahedron $\mathsf{Asso}(\mathrm{T})$, defined equivalently as

(i) the convex hull of the points

$$\mathbf{a}(\mathrm{S})_v = \begin{cases} \big| \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathrm{V}^- \\ |\mathrm{V}| + 1 - \big| \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathrm{V}^+ \end{cases}$$

for all maximal signed spines $\mathrm{S} \in \mathcal{S}(\mathrm{T})$

(ii) the intersection of the hyperplane $\mathbb{H}$ with the half-spaces

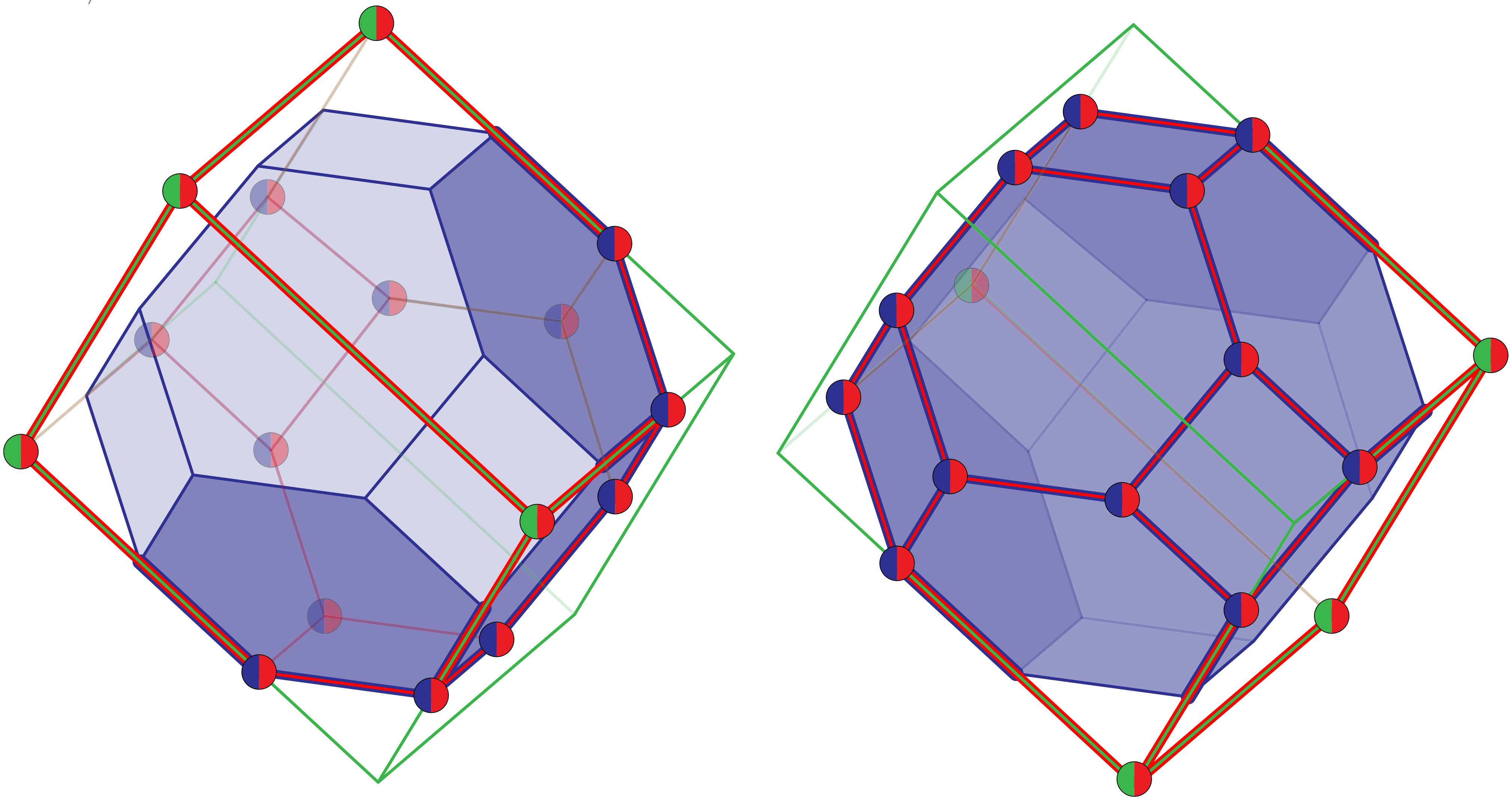
$$\mathbf{H}^{\geq}(B) := \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{V}} \;\middle|\; \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}$$

for all signed building blocks $B \in \mathcal{B}(\mathrm{T})$

SIGNED TREE ASSOCIAHEDRA

The signed tree associahedron Asso(T) is sandwiched between the permutahedron Perm(V) and the parallelepiped Para(T)

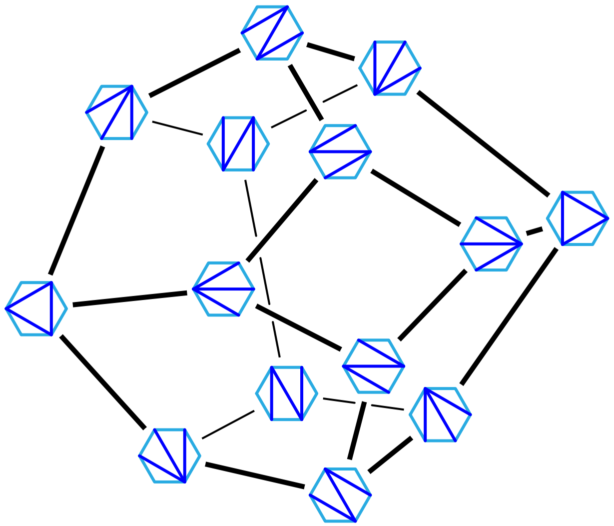
$$\sum_{u \neq v \in \mathrm{V}} [e_u, e_v] = \text{Perm}(\mathrm{T}) \quad \subset \quad \text{Asso}(\mathrm{T}) \quad \subset \quad \text{Para}(\mathrm{T}) = \sum_{u - v \in \mathrm{T}} \pi(u - v) \cdot [e_u, e_v]$$



WHAT SHOULD I TAKE HOME FROM THIS TALK?

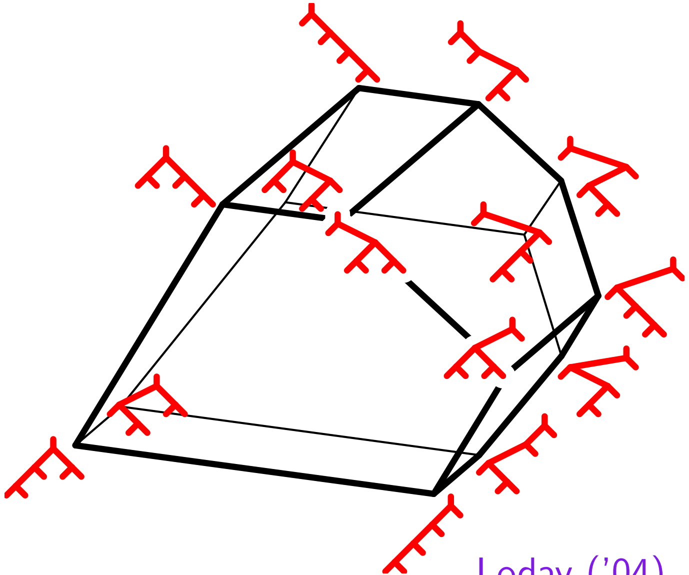
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

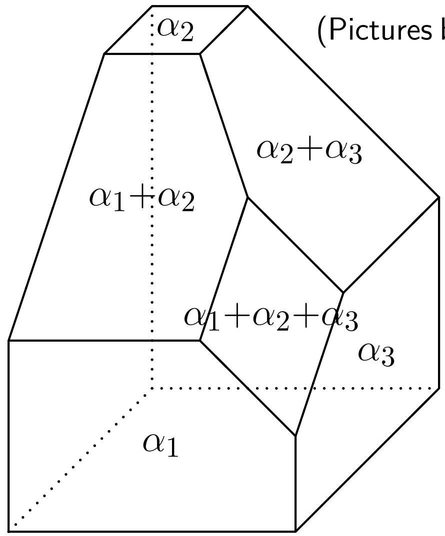
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

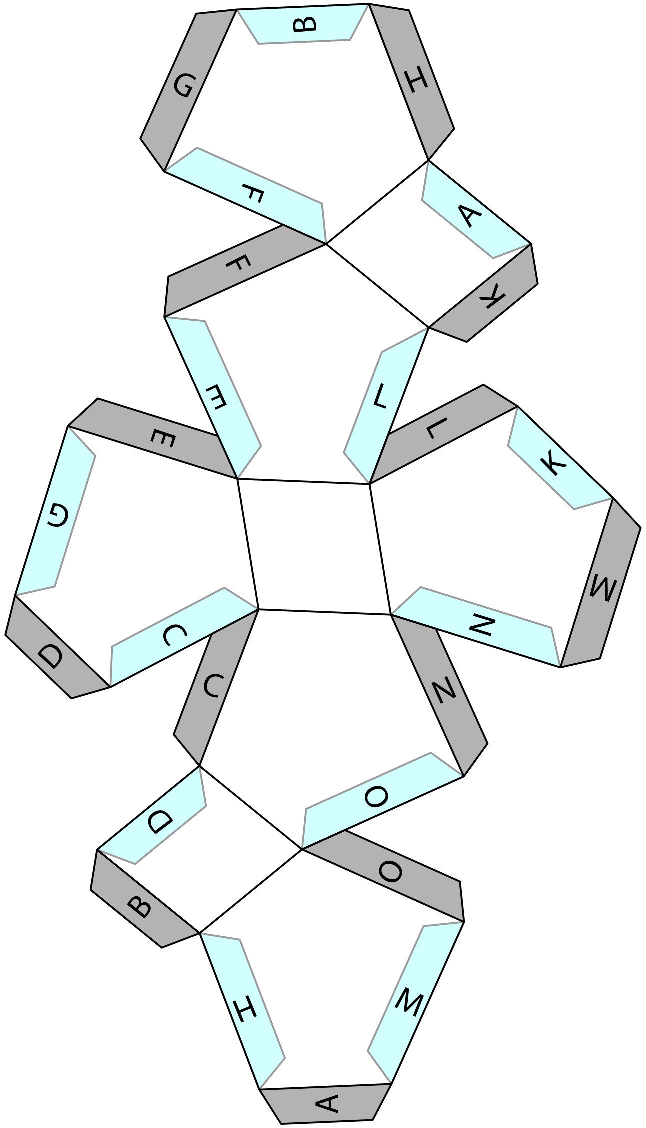
(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

TAKE HOME YOUR ASSOCIAHEDRA!

SECONDARY POLYTOPE



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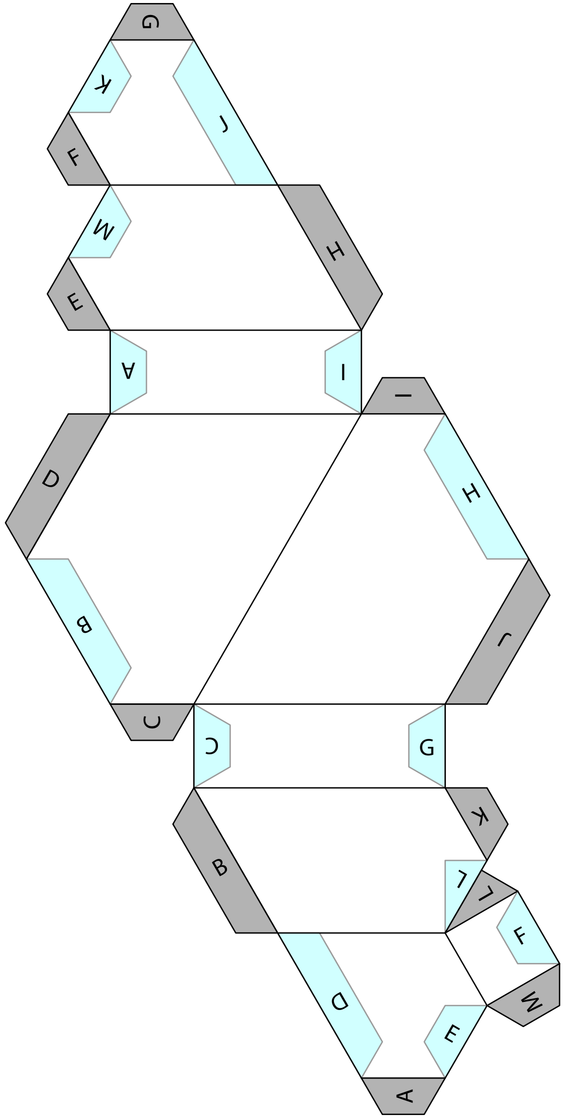
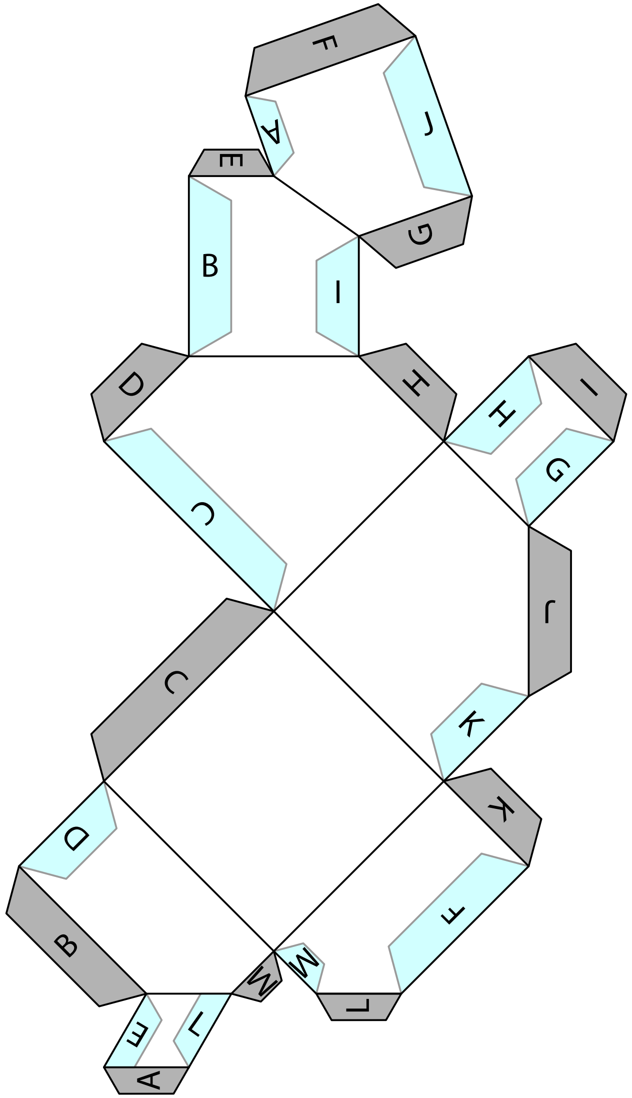
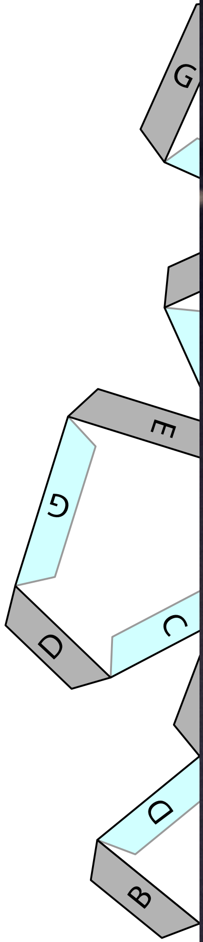
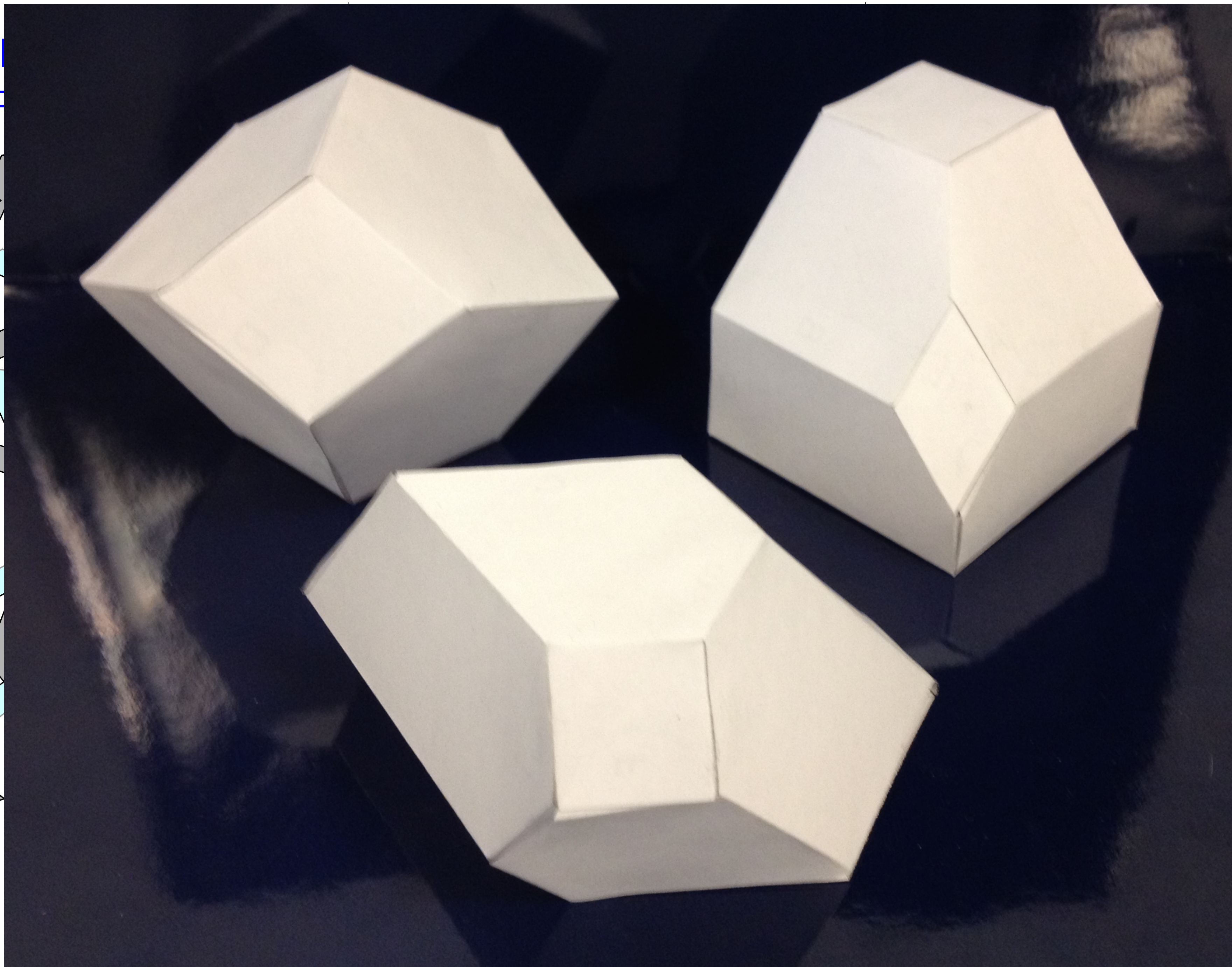


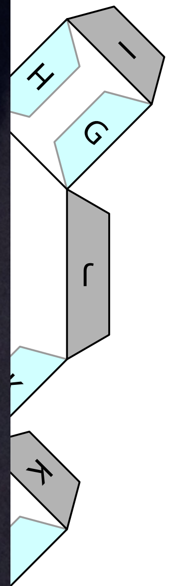
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON



TAKE HOME YOUR ASSOCIAHEDRA!



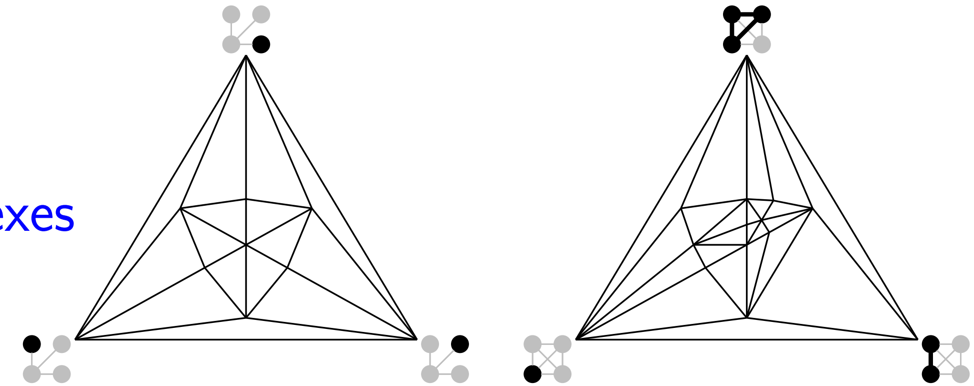




Thibault Manneville & VP

Compatibility fans for graphical nested complexes

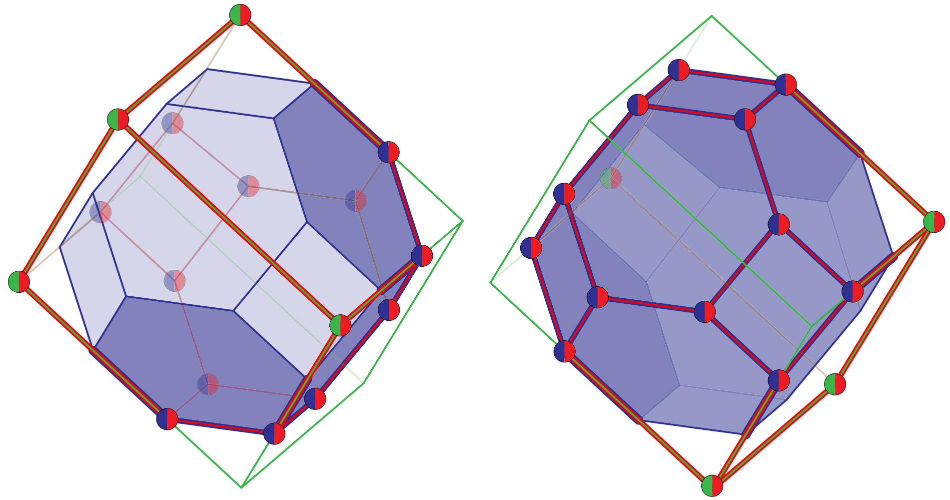
arXiv:1501.07152



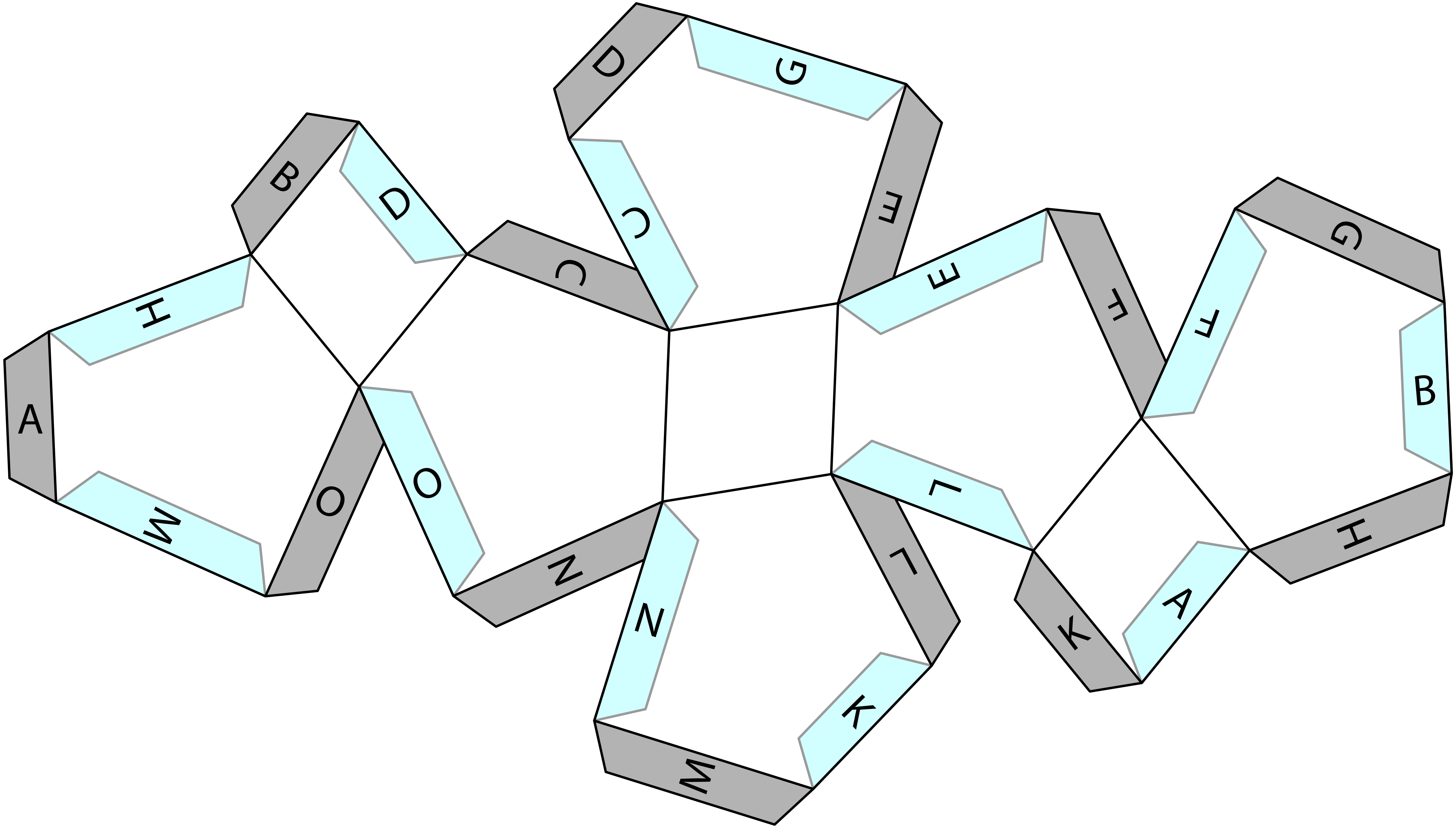
VP

Signed tree associahedra

arXiv:1309.5222



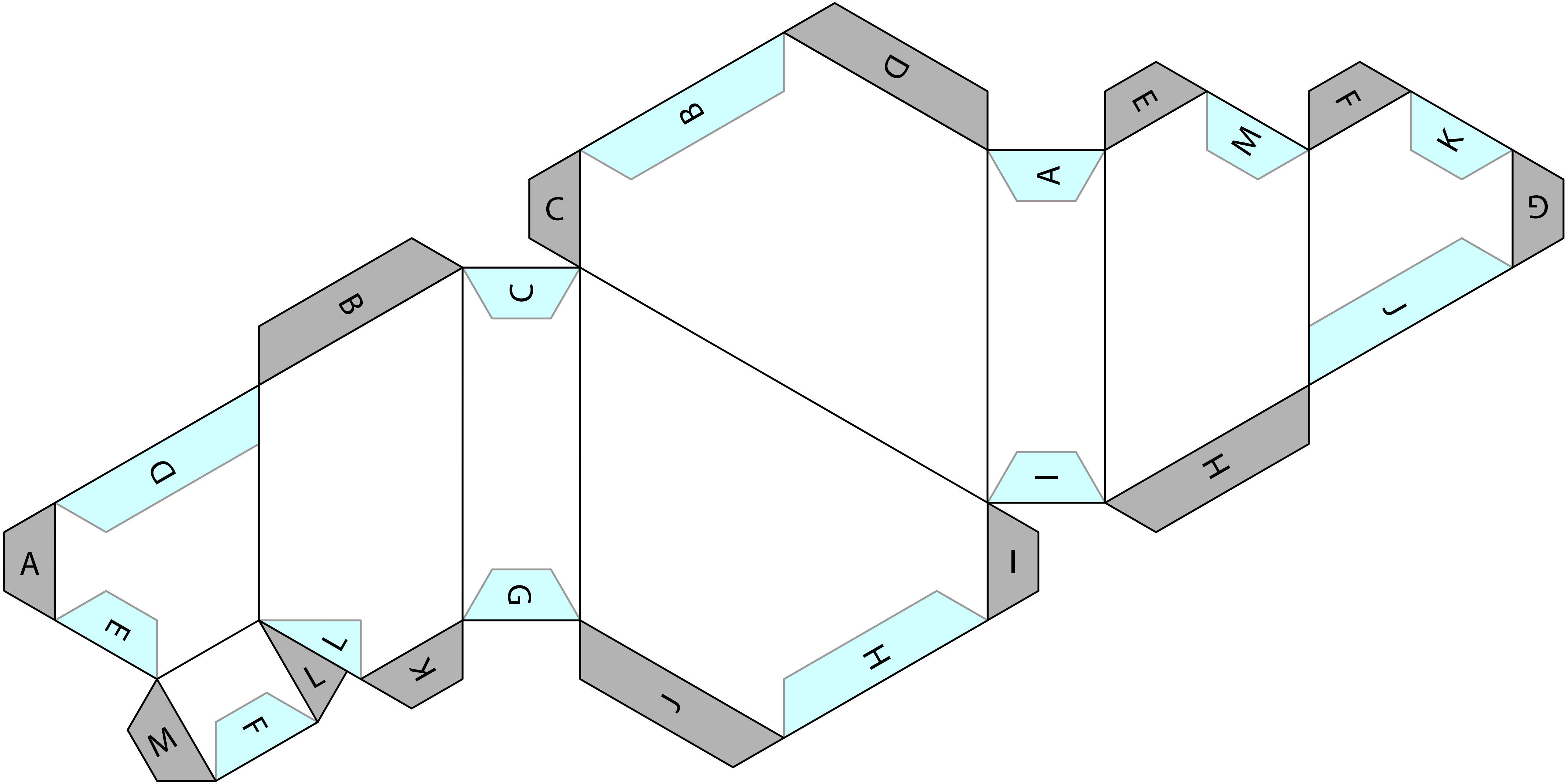
THANK YOU



SECONDARY POLYTOPE

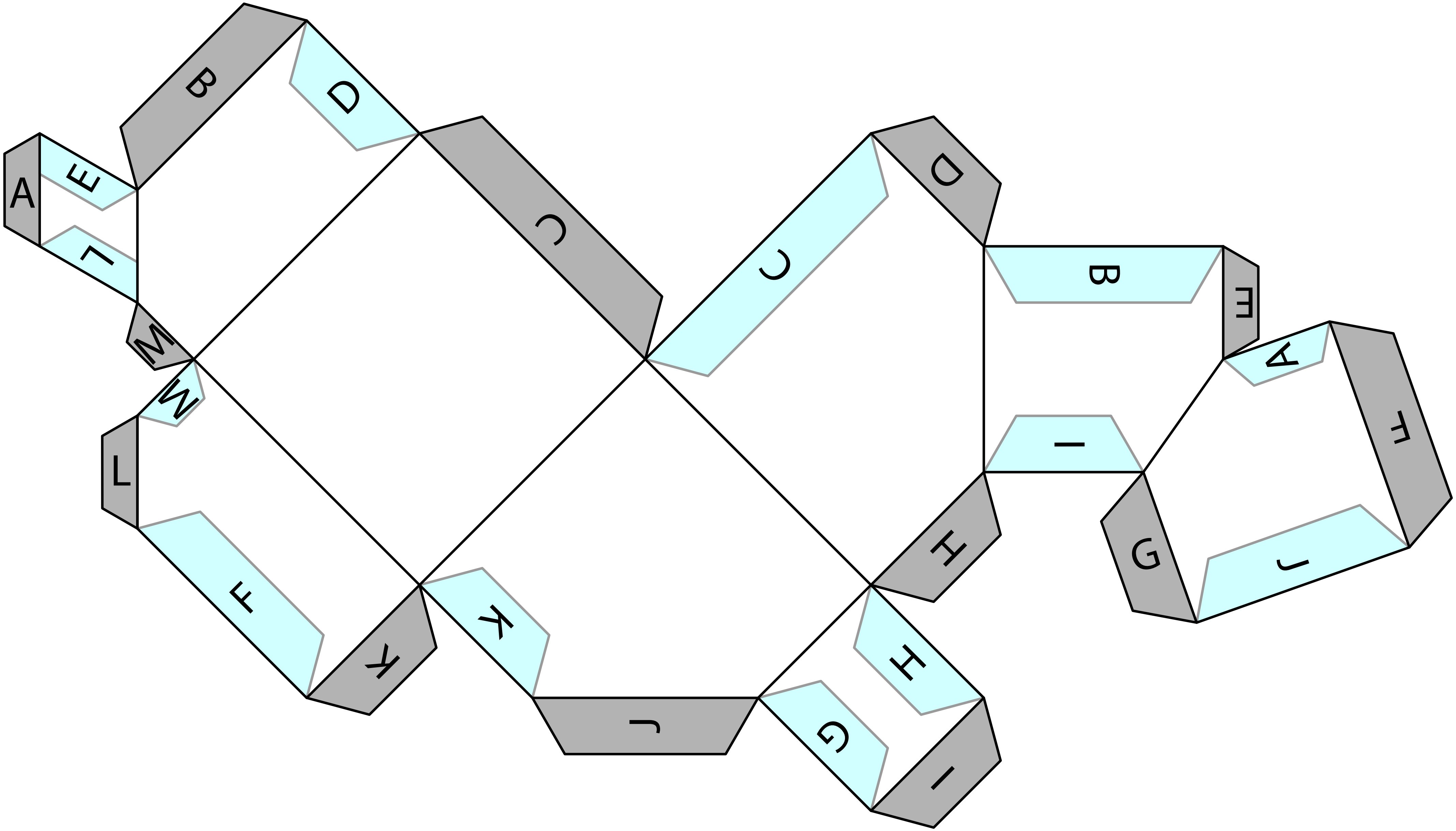
Gelfand-Kapranov-Zelevinsky ('94)

Billera-Filliman-Sturmfels ('90)



LODAY'S ASSOCIAHEDRON

Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)



CHAPOTON-FOMIN-ZELEVINSKY'S ASSOCIAHEDRON

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